M. Phys I

Mathematical Analysis 3

Solutions to Sheet 8.

1) a) $u_n = \dfrac{1}{(n+1)!}$ $\therefore$ $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)!}{(n+2)!} = \lim\limits_{n\to\infty}\dfrac{1}{n+2} = 0$. Convergent

b) $u_n = \dfrac{(3-4i)^n}{n!}$ $\therefore$ $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\dfrac{n!}{(n+1)!}\left|\dfrac{(3-4i)^{n+1}}{(3-4i)^n}\right| = \lim\limits_{n\to\infty}\dfrac{5}{n+1} = 0$. Convergent.

2) $\dfrac{dy}{dx} = \sec^2 x = 1 + \tan^2 x$. Since $\tan x$ is odd, we need to go to x^5.

Differentiate the diff. equation 4 times: $y'' = 2yy'$; $y''' = 2(yy'' + y'^2)$

$y^{(4)} = 2(yy''' + 3y'y'')$, $y^{(5)} = 2(yy^{(4)} + 4y'y''' + 3y''^2)$.

At $x = 0$: $y = 0$, $y' = 1$, $y'' = 0$, $y''' = 2$, $y^{(4)} = 0$, $y^{(5)} = 16$.

$$\therefore \quad y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \frac{x^5}{5!}y^{(5)}(0)\ldots$$

$$= 0 + x + 0 + \frac{2}{3!}x^3 + 0 + \frac{16}{5!}x^5\ldots$$

$$\therefore \quad y = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

3) $u_x = \dfrac{x^2 + 2xy + 2y - y^2 - 1}{(x+y)^2}$; $u_y = \dfrac{2xy + y^2 - x^2 - 2x - 1}{(x+y)^2}$

$u_x = u_y = 0$ Add to get $2xy = x - y + 1$. Subtract to get $x - y + 1 = 0$ $(x + y \neq 0)$.

Together we have $x = 0, y = 1$ and $y = 0, x = -1$. Two points $(0,1), (-1,0)$.

After a lot of work: $u_{xx} = \dfrac{4y^2 - 4y + 2}{(x+y)^3}$ $u_{yy} = \dfrac{4x^2 + 4x + 2}{(x+y)^3}$

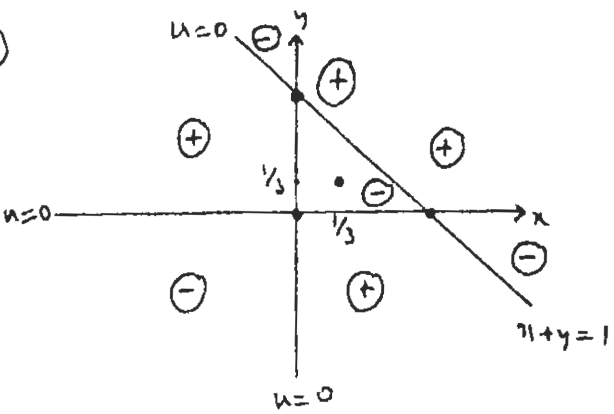
$u_{xy} = \dfrac{2(x - y - 2xy + 1)}{(x+y)^3}$

$(-1,0)$: $u_{xx} = -2$, $u_{yy} = -2$, $u_{xy} = 0$ MAX.

$(0,1)$ $u_{xx} = 2$, $u_{yy} = 2$, $u_{xy} = 0$ MIN.

(Note: At the max $u = 0$ while at the min $u = 2$. How can this be? Consider that u becomes infinite along the line $y = -x$.)

4)



Signs in circles refer to the sign of u.

$u_x = y(2x + y - 1)$, $u_y = x(x + 2y - 1)$

$\therefore$ Stat. pts at $(0,0), (0,1), (1,0)$ & $(\tfrac{1}{3}, \tfrac{1}{3})$.

Consider changes of sign in u across each point. Clearly $(0,0), (0,1), (1,0)$ are SADDLES.

Check $u_{xy}^2 - u_{xx}u_{yy} > 0$. This is 1 for these points, and is $-1/3$ for $(1/3, 1/3)$. This is a MINIMUM as $u_{xx} > 0$ here.