M. Phys 1. Mathematical Analysis

Solutions to Sheet 7.

1) $V(x,y) = f(x+y) + g(x+\tfrac{1}{2}y)$

Let $u = x+y \quad v = x+\tfrac{1}{2}y$ $\qquad u_x = 1 \quad u_y = 1$, $\quad v_x = 1 \quad v_y = \tfrac{1}{2}$

so $V = f(u) + g(v)$

$$\therefore \frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial V}{\partial u} + \frac{\partial V}{\partial v} = f' + g' \qquad 1)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial V}{\partial u} + \frac{1}{2}\frac{\partial V}{\partial v} = f' + \tfrac{1}{2}g' \qquad 2)$$

Note that 1) implies that the derivative operation $\partial/\partial x$ can be written as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{and similarly} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(f'+g') = f'' + g''$$

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}\right)(f' + \tfrac{1}{2}g') = f'' + \tfrac{1}{4}g''$$

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(f' + \tfrac{1}{2}g') = f'' + \tfrac{1}{2}g''$$

Hence $V_{xx} + 2V_{yy} = 3V_{xy}$

2) $s = \dfrac{x}{x^2+y^2} \qquad t = \dfrac{y}{x^2+y^2}$

$$\frac{\partial s}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad \frac{\partial s}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial t}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial t}{\partial y} = \frac{-y^2+x^2}{(x^2+y^2)^2}$$

Chain rule:

$$u_x = u_s\frac{\partial s}{\partial x} + u_t\frac{\partial t}{\partial x} = [u_s(y^2-x^2) - 2xy\,u_t](x^2+y^2)^{-2}$$

$$u_y = u_s\frac{\partial s}{\partial y} + u_t\frac{\partial t}{\partial y} = [-2xy\,u_s + (x^2-y^2)u_t](x^2+y^2)^{-2}$$

$$\therefore (u_x^2 + u_y^2)(x^2+y^2)^4 = u_s^2(x^2+y^2)^2 + u_t^2(x^2+y^2)^2$$

Hence result.

3) Write the differential as $P\,dx + Q\,dy$.

To be able to write this as df and find $f(x,y)$ we need (notes) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

i) $P = e^y, \; Q = x(e^y+1)$ $\quad$ Clearly $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ No!

ii) $P = e^y + ye^x, \; Q = e^x + e^y + 1$

$\dfrac{\partial P}{\partial y} = e^y + e^x, \quad \dfrac{\partial Q}{\partial x} = e^x + e^y.$ Yes! $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$\underline{f = xe^y + y(e^x+1)}$