Physics I Mathematical Analysis Solutions 6

1) $\frac{\partial u}{\partial x} = 8xy+3 \qquad \frac{\partial u}{\partial y} = 4x^2 - 2y$

2) i) $u = e^{\alpha x}\cos\beta y \qquad \frac{\partial u}{\partial x} = \alpha e^{\alpha x}\cos\beta y, \quad \frac{\partial^2 u}{\partial x^2} = \alpha^2 e^{\alpha x}\cos\beta y$

$\frac{\partial u}{\partial y} = -\beta e^{\alpha x}\sin\beta y, \quad \frac{\partial^2 u}{\partial y^2} = -\beta^2 e^{\alpha x}\cos\beta y$

Need $\alpha^2 - \beta^2 = 0$ to satisfy Laplace's eqn. $\Rightarrow \alpha = \pm\beta$.

ii) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x, \quad \frac{\partial^2 u}{\partial x^2} = 6x+6$

$\frac{\partial u}{\partial y} = -6xy - 6y \qquad \frac{\partial^2 u}{\partial y^2} = -6x-6$

Sum is zero.

3) $g = \tan^{-1}(y/x) \quad \therefore \frac{\partial g}{\partial x} = \frac{\frac{\partial}{\partial x}(y/x)}{1+(y/x)^2} = \frac{-y/x^2}{1+y^2/x^2} = \frac{-y}{x^2+y^2}$

$\frac{\partial g}{\partial y} = \frac{\frac{\partial}{\partial y}(y/x)}{1+(y/x)^2} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}$. Hence $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 0$

4) i) Chain Rule: $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$

$x = s+t \qquad \frac{\partial x}{\partial s} = 1 \quad \frac{\partial x}{\partial t} = 1$

$y = 2s-t \qquad \frac{\partial y}{\partial s} = 2 \quad \frac{\partial y}{\partial t} = -1$

$= \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}$

$= 2x + 2.(9y^2)$

$= 2(s+t) + 18(2s-t)^2$

$= 18(4s^2 - 4st + t^2) + 2s + 2t \qquad (*)$

Now $u = x^2 + 3y^3 = (s+t)^2 + 3(2s-t)^3 = s^2 + 2st + t^2$
$+ 3(8s^3 - 12s^2t + 6st^2 - t^3)$

so $\frac{\partial u}{\partial s} = 2s + 2t + 3(24s^2 - 24st + 6t^2)$ Same as $(*)$

Do the same for $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$ etc.

5) $V = xyz$ (V fixed) $\quad A = 2(xy + yz + xz)$

Eliminate z in A using $z = V/xy$.

$\therefore \quad A = 2(xy + \frac{V}{x} + \frac{V}{y}) \qquad V$ constant.

$\therefore \quad \frac{\partial A}{\partial x} = 2\left(y - \frac{V}{x^2}\right), \quad \frac{\partial A}{\partial y} = 2\left(x - \frac{V}{y^2}\right)$

For $A_x = 0$ & $A_y = 0$ together we have

$V = x^2y$ & $V = xy^2$ with $V = xyz$. Only solution

is $x = y = z = V^{1/3}$. Minimum by inspection.