PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 6

1. Calculate $\partial u/\partial x$ and $\partial u/\partial y$ if $u = 4x^2y - y^2 + 3x - 1$.

2. Find the relation between the constants α and β if the function $u = e^{\alpha x}\cos\beta y$ satisfies Laplace's equation:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$
Show also that the following function $u(x, y)$ satisfies Laplace's equation
$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.$$

3. If $g = \tan^{-1}(y/x)$, calculate $\partial g/\partial x$ and $\partial g/\partial y$, and show that
$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 0$$

4. If $u = x^2 + 3y^3$ and $x = s + t, y = 2s - t$, calculate $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ (i) by using the chain rule, and (ii) by first expressing u as a function of s and t.

5. A closed box has variable sides of length x, y and z but a fixed volume V. Show that the shape of the box is a cube when the surface area A is minimum. *Note:* at a stationary point of a function of two variables $a = a(x, y)$ the two partial derivatives a_x and a_y need to be zero simultaneously.

STARRED PROBLEMS

6* If $u = x\ln(x^2 + y^2) - 2y\tan^{-1}(y/x)$, verify that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + 2x.$$

7* The equation of state[1] of a gas, relating pressure p, volume V and temperature T, is $f(p, V, T) = 0$ and hence
$$df = \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial V}dV + \frac{\partial f}{\partial T}dT = 0.$$
Show that
$$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{(\partial f/\partial V)_{p,T}}{(\partial f/\partial p)_{V,T}}$$
and obtain similar expressions for $(\partial V/\partial T)_p$ and $(\partial T/\partial p)_V$. Deduce that
$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = -1.$$

[1] In this problem we need not specify the function f; it is left as an arbitrary function of the three independent variables p, V and T but for an ideal gas it would take the form $f = pV - RT = 0$. In fact, you can verify some of the above formulae using this relation.