PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 5

Recall from your notes that:

(i) Plane polar co-ordinates (r, θ) are related to Cartesian co-ordinates (x, y) by $x = r\cos\theta$ and $y = r\sin\theta$: hence $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
(ii) In Cartesian co-ordinates a small element of arc length ds is related to the small elements dx and dy by $(ds)^2 = (dx)^2 + (dy)^2$ (with an additional $(dz)^2$ in 3D). In plane polar co-ordinates this converts to $(ds)^2 = (dr)^2 + r^2(d\theta)^2$.
(iii) Volume of revolution is $\pi\int_a^b y^2\,dx$ whose surface area is $2\pi\int_a^b y\,ds$.

1. Find the lengths of the following curves:

   (a) The catenary $y = \cosh x$ from $x = 0$ to $x = 1$. [Answer: $\sinh 1$].

   (b) The circular helix expressed in parametric form $x = \cos t$, $y = \sin t$ and $z = t$ from $t = 0$ to $t = 2\pi$. [Answer: $2\sqrt{2}\pi$].

   (c) The curve $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$. [Answer: $\frac{8}{27}\left(10^{3/2} - 1\right)$].

   (d) One branch of the 4-cusped hypocycloid expressed in parametric form $x = \cos^3 t$, $y = \sin^3 t$ from $t = 0$ to $t = \pi/2$. [Answer: $3/2$].

2. Show that the area of one loop $(-\pi/4 \le \theta \le \pi/4)$ of the lemniscate $r^2 = a^2\cos 2\theta$ is $a^2/2$.

3. Find the position of the centre of mass of a uniform thin wire in the form of a circular arc of radius a, subtending an angle of 2γ at the centre. [Answer: $a\sin\gamma/\gamma$ from the apex.]

4. Find the area enclosed by the ellipse $(x/a)^2 + (y/b)^2 = 1$. Assuming this elliptical area to be of uniform density, find also the position of the centre of gravity of the part that lies in the first quadrant. [Answers: πab and $(4a/3\pi, 4b/3\pi)$.]

STARRED PROBLEMS

5* Show that $8a$ is the total length of the closed curve called the cardioid (heart shape)

$$r = a(1 - \cos\theta).$$

6* Show that the length of one arch $(0 \le t \le 2\pi)$ of the cycloid defined by

$$x = a(t - \sin t) \qquad y = a(1 - \cos t)$$

is $8a$. Show that the area of the surface obtained by a complete revolution of this arch about the x-axis is $64\pi a^2/3$.