MPhys I

Mathematical Analysis

Solutions to Problems 3.

① a) $\int x^3 \sin x\,dx = -\int x^3 d(\cos x) = -x^3\cos x + 3\int x^2 \cos x\,dx$

$= -x^3\cos x + 3\int x^2 d(\sin x)$

$= -x^3 \cos x + 3x^2\sin x - 6\int x \sin x\,dx$

$= -x^3\cos x + 3x^2 \sin x + 6\int x\,d(\cos x)$

$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + c$

b) $\int \underbrace{\tan^{-1}x}_{u}\,\underbrace{dx}_{v} = x\tan^{-1}x - \int x \frac{d}{dx}(\tan^{-1}x)\,dx$

$= x\tan^{-1}x - \int \frac{x\,dx}{1+x^2} = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c.$

② a) Use $x = \tan\theta$ $\int_0^1 \frac{dx}{(1+x^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/4}\cos\theta\,d\theta = \frac{1}{\sqrt{2}}$

b) $\int_0^\infty \frac{dx}{1+e^{2x}} = \frac{1}{2}\int_1^\infty \frac{du}{u(1+u)} = \frac{1}{2}\int_1^\infty\left(\frac{1}{u} - \frac{1}{1+u}\right)du = \frac{1}{2}\left[\ln\left|\frac{u}{1+u}\right|\right]_1^\infty$

$= \frac{1}{2}\{\ln 1 - \ln\frac{1}{2}\} = \frac{1}{2}\ln 2.$

c) $\int_1^{3/2} \frac{dx}{(2-x)(x-1)^{1/2}} = \int_0^{1/\sqrt{2}} \frac{2\cancel{u}\,du}{(1-u^2)\cancel{u}} = \int_0^{1/\sqrt{2}}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)du$

$= \left[\ln\left|\frac{1+u}{1-u}\right|\right]_0^{1/\sqrt{2}} = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\ln(\sqrt{2}+1).$

③ a) $\int_\varepsilon^1 \ln x\,dx = [x(\ln x - 1)]_\varepsilon^1 = -1 - \varepsilon(\ln\varepsilon - 1) = \varepsilon - 1 - \varepsilon\ln\varepsilon$

What does $\varepsilon\ln\varepsilon$ do as $\varepsilon\to 0$? Actually $\lim_{\varepsilon\to 0}(\varepsilon\ln\varepsilon)\to 0$, so

RHS $\to -1$ as $\varepsilon \to 0$. ∴ Convergent.

b) $\int_0^{1-\varepsilon}\frac{dx}{(x-1)^2} = -[(x-1)^{-1}]_0^{1-\varepsilon} = \frac{1}{\varepsilon} - 1$. Not convergent as $\varepsilon\to 0$.

④ $\int x^k \ln x\,dx = \frac{1}{k+1}\int \ln x\,d(x^{k+1}) = \frac{1}{k+1}\left[(\ln x)x^{k+1} - \int x^{k+1}x^{-1}dx\right]$

$= \frac{1}{k+1}\left[x^{k+1}\ln x - \frac{1}{k+1}x^{k+1}\right] + c$ $\qquad k \neq -1.$

$= (k+1)^{-2}[(k+1)\ln x - 1]\,x^{k+1} + c.$