1. (a) Define what it means to say that a sequence (a_n) converges to a limit l.

   (b) Let (a_n) be a convergent sequence and (b_n) be a bounded sequence. Prove, directly from your definition, that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n b_n^2 = 0$.

   (c) State the Bolzano-Weierstrass theorem.

   (d) Define what it means for a function $f : [a,b] \to \mathbf{R}$ to be continuous at a point $c \in (a,b)$ and what it means for a function f to be continuous on the interval $[a,b] \subset \mathbf{R}$.

   (e) Using the Bolzano-Weierstrass theorem, or otherwise, prove that if $f : [a,b] \to \mathbf{R}$ is a continuous function, then it attains its least upper bound on $[a,b]$. In other words, there exists $c \in [a,b]$ such that

$$f(c) = \mathrm{lub}\{f(x) \mid x \in [a,b]\}.$$

   [*You can use without proof, the fact that since f is continuous on $[a,b]$, it is bounded on $[a,b]$.*]

2. (a) State the Monotonic Sequence Theorem.

   (b) Show that

$$\left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

   for all $n \geq 1$.

   (c) Using (b), or otherwise, show that $\left(1+\frac{1}{n}\right)^n < n$ for all $n \geq 3$.

   (d) Let (a_n) be the sequence defined as $a_n = n^{\frac{1}{n}}$ for all $n \geq 1$. Using (c), or otherwise, show that $a_{n+1} < a_n$ for all $n \geq 3$.

   (e) Prove that (a_n) is a convergent sequence.

   (f) Prove that $a_{2n} = 2^{\frac{1}{2n}}\sqrt{a_n}$ for all $n \geq 1$ and find the limit of (a_n).

   [*If required, you can use without proof the limit $\lim_{n\to\infty} 2^{\frac{1}{2n}} = 1$ and the fact that the square root function is continuous.*]

3. (a) Define what it means to say that a sequence (d_n) is Cauchy and state the General Principle of Convergence.

(b) Let (d_n) be a sequence for which there exists a constant $C \in (0,1)$ such that

$$|d_{n+2} - d_{n+1}| \leq C\,|d_{n+1} - d_n|$$

for all $n \geq 1$. Show that, for any positive integers m, n such that $m < n$, we have

$$|d_m - d_n| \leq \frac{C^{m-1}}{1-C}\,|d_2 - d_1|\,.$$

(c) Using (b) or otherwise, prove that (d_n) is convergent sequence.

(d) Let (e_n) be a sequence such that

$$e_{n+2} = \frac{1}{3}e_{n+1} + \frac{2}{3}e_n.$$

for all $n \geq 1$. Using (c) or otherwise, prove that (e_n) is convergent sequence.

(e) Find the limit of the sequence (e_n).

4. (a) Define what it means to say that a series $\sum a_n$ is convergent.

(b) Determine whether the series $\sum a_n$ converges in each of the following cases:

(i) $$a_n = \frac{1}{2n} - \frac{1}{2n+1}$$

(ii) $$a_n = 2008\left(\frac{1}{n} - \frac{1}{n+1}\right)\sin\left(\frac{(2n+1)\pi}{2}\right)$$

(iii) $$a_n = \frac{n}{\sqrt{3n^2+1}}$$

[*Give reasons in each case. You may use any standard tests and results without proof, provided that you make it clear which ones you are using.*]

(c) Define the radius of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$.

(d) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(3-4i)}{n^3} z^n$.

(e) Prove that the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(3-4i)}{n^3} z^n$ converges for any $z \in \mathbb{C}$ such that $|z| = \frac{1}{2}$.