1. (a) Define what it means to say that a sequence (a_n) converges to a limit l.

(b) Let (a_n) be a sequence with non-negative terms.

(i) Prove, directly from your definition, that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \sqrt{a_n} = 0$.

(ii) Prove, directly from your definition, that if $\lim_{n\to\infty} a_n = 2007$, then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{2007}.$$

(c) Let (b_n) be a convergent sequence of complex numbers and (c_n) be another sequence such that

$$|b_n - c_n| \leq \frac{n}{n\sqrt{n} - 1}$$

for all $n \geq 2$. Prove that (c_n) is a convergent sequence. [*You may use any standard results without proofs, provided that you make it clear which ones you are using.*]

2. (a) State the Monotonic Sequence Theorem.

(b) Let (c_n) be the sequence defined as

$$c_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$$

for all $n \geq 1$.

(i) Prove that $\frac{1}{2} \leq c_n \leq 1$ for any $n \geq 0$.

(ii) Prove that (c_n) is a convergent sequence.

(c) Define what it means to say that a sequence (d_n) is Cauchy and state the General Principle of Convergence.

(d) Let (e_n) and (f_n) be two convergent sequences and (g_n) be the sequence defined as

$$g_n = \begin{cases} e_n & \text{if } n \text{ odd} \\ f_n & \text{if } n \text{ even} \end{cases}$$

for all $n \geq 1$. Show that (g_n) is Cauchy if and only if $\lim_{n\to\infty} e_n = \lim_{n\to\infty} f_n$.

3. (a) Define what it means to say that a series $\sum a_n$ is convergent.

   (b) Determine whether the series $\sum a_n$ converges in each of the following cases:

   (i) $a_n = \dfrac{\cos \pi n}{2\sqrt{n} - 1}$

   (ii) $a_n = \dfrac{(3n)!}{(n!)^3}$

   (iii) $a_n = \dfrac{n\sqrt{n} - 1}{n^2 + 1}$

   (iv) $a_n = (n+2)^{\frac{1}{n}}$.

   [*Give reasons in each case. You may use any standard tests and results without proof, provided that you make it clear which ones you are using.*]

4. (a) Define the radius of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$.

   (b) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (1+i)}{n} z^{2n}$

   (c) Prove that the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (1+i)}{n} z^{2n}$ diverges for $z = -\frac{1}{\sqrt{3}}$.

   (d) Define the product of two powers series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n z^n$.

   (e) Find the product of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n!} z^{2n}$ with itself.

5. (a) Define what it means for a function $f : \mathbf{R} \to \mathbf{R}$ to be continuous at a.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be two functions continuous at $a \in \mathbf{R}$. Show that if $f(a) > g(a)$ then there exists $\delta > 0$ such that $f(x) > g(x)$ for any $x \in (a - \delta, a + \delta)$.

(c) Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be two continuous functions. Show that the function $h : \mathbf{R} \to \mathbf{R}$ defined as

$$h(x) = \max(f(x), g(x))$$

for any $x \in \mathbf{R}$, is continuous.