Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M1P1

Analysis I

Date: Monday, 8th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Define what it means to say that a sequence (a_n) converges to a limit l.

(b) Prove (directly from your definition) that

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n-1}\right)=0.$$

(c) State the Monotonic Sequence Theorem.

(d) Let (a_n) be the sequence that satisfies the identity

$$a_{n+1}=\frac{1}{4}\left(1+a_n\right)$$

for all $n\geq 1$ and assume that $a_1\leq\frac{1}{3}$.

(i) Prove that $a_n\leq\frac{1}{3}$ for all $n\geq 1$.

(ii) Prove that (a_n) is convergent and find its limit.

2. (a) Define what it means to say that a sequence (a_n) is Cauchy.

(b) State the General Principle of Convergence.

(c) Let (a_n) be the sequence defined as

$$a_n=\sum_{k=1}^{n}\frac{1}{k^3}\quad\text{for all } n\geq 1.$$

Using the inequality $\frac{1}{k^2}<\frac{1}{k-1}$ for $k>1$, or otherwise, prove that

$$|a_m-a_n|<\frac{1}{n}-\frac{1}{m}$$

for any $1<n<m$.

(d) Show that the sequence (a_n) as defined in (c) is a convergent sequence.

[*You can use without proof the fact that* $\frac{1}{n}\to 0$.]

3. (a) Define what it means to say that a series $\sum a_n$ is convergent.

(b) Determine whether the series $\sum a_n$ converges in each of the following cases:

(i) $$a_n = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \ ,$$

(ii) $$a_n = \frac{2+\cos n}{1+3n} \ ,$$

(iii) $$a_n = (-1)^n \left(\frac{1}{5n+2} + \frac{1}{5n-3} \right),$$

(iv) $$a_n = \frac{2^n}{3^n + 4^n} \ .$$

[*Give reasons in each case. You may use any standard tests and results without proof, provided that you make it clear which ones you are using.*]

4. (a) Define the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$.

(b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{n+1} z^n$.

(c) Define the product of two powers series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$.

(d) Using the identity

$$\frac{1}{ab} = \frac{1}{a+b}\left(\frac{1}{a} + \frac{1}{b}\right),$$

where a and b are non-zero constants, show that the product of the power series $\sum_{n=0}^{\infty} \frac{1}{n+1} z^n$ with itself is

$$\sum_{n=0}^{\infty} \frac{2}{n+2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n+1}\right) z^n.$$

(e) Let (a_n) be a sequence with positive terms such that $(a_n)^{\frac{1}{n}} \to L$, where L is a positive constant. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is $\frac{1}{L}$.

5. (a) Define what it means for a function $f : \mathbf{R} \to \mathbf{R}$ to be continuous at a.

(b) Let $g : \mathbf{R} \to \mathbf{R}$ be a bounded function. Prove (directly from your definition) that the function $f : \mathbf{R} \to \mathbf{R}$ defined as

$$f(x) = \left(x^2 - 1\right) g(x)$$

is continuous at 1.

(c) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $f(2x) = f(x)$ for any $x \in \mathbf{R}$.

(i) Prove that $f(x) = f\left(\frac{x}{2^n}\right)$ for any $x \in \mathbf{R}$ and $n \geq 0$.

(ii) Using the fact that $\frac{1}{2^n} \to 0$ prove that f is a constant function.