1. (a) Define what it means to say that a sequence (a_n) converges to a limit l.

(b) Prove (directly from your definition) that

(i) $\dfrac{\cos(n\pi)}{\sqrt{n}} \to 0.$

(ii) $\dfrac{n^2 + (-1)^n n}{3 + 2n^2} \to \dfrac{1}{2}$.

(c) Let (a_n) be a sequence with nonnegative terms and (b_n) be the sequence defined as

$$b_n = \frac{a_n - 1}{a_n + 1}$$

for all $n > 0$. Prove that $b_n < 1$ for all $n > 0$ and that

$$|a_n - 1| = \frac{2\,|b_n|}{1 - b_n} \ .$$

(d) Use (c) or otherwise to show that $b_n \to 0$ implies $a_n \to 1$ (again, directly from your definition).

2. (a) Define what it means to say that a sequence (a_n) is Cauchy.

(b) State the General Principle of Convergence.

(c) Let (a_n) be a sequence with the property that there exists a positive constant $r < 1$ such that

$$|a_{n+1} - a_n| \le r|a_n - a_{n-1}|$$

for all $n > 1$. Prove that, for any $0 < n < m$,

$$|a_m - a_n| \le \left(r^n + r^{n+1} + \ldots + r^{m-1}\right)|a_1 - a_0| \, .$$

(d) Using (c) or otherwise show that (a_n) is a convergent sequence.
[*You can use without proof the fact that* $r^n \to 0$ *if* $0 \le r < 1$.]

3. (a) Define what it means to say that a series $\sum a_n$ is convergent.

(b) Determine whether the series $\sum a_n$ converges in each of the following cases:

(i) $a_n = \left(\cos\left(\dfrac{n\pi}{2}\right)\right)^2$

(ii) $a_n = \dfrac{2n+1}{n+n^2}$

(iii) $a_n = \dfrac{(-1)^n}{n\ln(n+1)}$

(iv) $a_n = \dfrac{2 + (-1)^n}{4^n}$.

[*Give reasons in each case. You may use any standard tests and results without proof, provided that you make it clear which ones you are using.*]

4. (a) Define the product of two powers series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$.

(b) Show that the product of the power series $\sum_{n=0}^{\infty} (-1)^n z^{2n}$ with itself is

$$\sum_{n=0}^{\infty} (-1)^n (n+1) z^{2n}.$$

(c) Define the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$.

(d) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n (n+1) z^{2n}.$$

5. (a) Show that the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

is absolutely convergent for any real number x.

(b) Show that

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \leq \sum_{n=1}^{\infty} x^{2n}$$

for $|x| < 1$.

(c) Define what it means for a function $f : \mathbf{R} \to \mathbf{R}$ to be continuous at a.

(d) Let f be the function defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

for any real number x. Use (b) to show that

$$|f(x) - 1| \leq \frac{x^2}{1 - x^2} \qquad (1)$$

for $|x| < 1$, and use (**??**) or otherwise to prove that f is continuous at 0.