M1P1

1

a Define what it means to say that a sequence (a_n) converges to a limit l.

b Prove (directly from your definition) that

i $\dfrac{(-1)^n}{n^2} \to 0$.

ii $\dfrac{1-2e^n}{e^n+3} \to -2$.

c Let (a_n) be a sequence convergent to 0 and (b_n) be the sequence defined as

$$b_n = \frac{1}{n}\left(a_1 + a_2 + ... + a_n\right) \text{ for all n} \geq 1.$$

Prove that $b_n \to 0$.

2

a State the Monotonic Sequence Theorem.

b Prove that the sequence (a_n) defined as $a_1 = 0.5$ and

$$a_{n+1} = a_n - a_n^2 \text{ for all n} \geq 1.$$

is convergent and find its limit.

c Let (a_n) be a sequence with positive terms ($a_n > 0$ for all $n > 0$). Suppose there exists a number $r < 1$ and an integer n_0 such that

$$\frac{a_{n+1}}{a_n} \leq r \text{ for all n} \geq \text{n}_0.$$

Prove that $a_n \longrightarrow 0$. [You can use without proof the result $r^n \longrightarrow 0$.]

d Using **(c)**, or otherwise, show that $\dfrac{2^n}{n!} \to 0$.

3

a Define what it means to say that a sequence (a_n) is Cauchy.

b State the General Principle of Convergence.

c Let (a_n) be the following sequence

$$a_n = \frac{1}{1} + \frac{1}{3} + ... + \frac{1}{2n-1} \text{ for all n} \geq 1.$$

Prove that $|a_{2n} - a_n| \geq \dfrac{n}{4n-1}$.

d Using **(c)**, or otherwise, show that (a_n) is not a convergent sequence.

4

a Define what it means to say that a series $\sum a_n$ is convergent.

b Determine whether the series $\sum a_n$ converges in each of the following cases:

i $a_n = \dfrac{1}{3n+1} - \dfrac{1}{3n+4}$.

ii $a_n = \dfrac{1+n}{n^2+1}$.

iii $a_n = (-1)^n \left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + ... + \dfrac{1}{2n} \right)$.

[Give reasons in each case. You may use any standard tests and results without proof, provided that you make it clear which ones you are using.]

c Let (a_n) and (b_n) be two sequences of positive terms such that $\dfrac{a_n}{b_n} \to 2$. Prove that $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent.

5

a Define the radius of convergence of a power series $\sum a_n z^n$ b Find the radius of convergence of the following power series:

i $\sum (n+1) z^n$.

ii $\sum 2^n z^{2n+1}$.

c Let (a_n) be a sequence for which there exists two positive constants r and a such that

$$\frac{|a_n|}{r^n} \to a.$$

Prove that the power series $\sum a_n z^n$ has radius of convergence $\dfrac{1}{r}$.

d What is the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1 - n2^n}{n+1} z^n$?