M1P1

1.
i. Define what it means to say that a sequence (a_n) *converges* to a limit l.
ii. Prove (directly from your definition) that
(a) $\frac{1}{\sqrt{n}} \to 0$; (b) $\frac{3n+1}{2n-1} \to \frac{3}{2}$;
iii. Let (a_n) be a bounded sequence and (b_n) be a sequence converging to 0. Prove that $a_n b_n \to 0$. Hence prove that

$$\left(\sqrt{n+2} - \sqrt{n}\right) \cos n \to 0.$$

2.
i. Let (a_n) be a sequence and (a_{n_k}) be a subsequence of (a_n). Prove that $a_n \to l$ implies $a_{n_k} \to l$.
ii. Using **i.** (or otherwise) show that the sequence (a_n), where

$$a_n = \frac{\sin n\pi + 2}{3 - \cos n\pi} \text{ for all } n > 0,$$

is not convergent.
iii. Let (a_n) be a sequence such that the subsequences (a_{2n}) and (a_{2n+1}) both converge to the same limit l. Prove that $a_n \to l$.
iv. Using **iii.** (or otherwise) show that

$$\frac{(-1)^n \sqrt{n}}{1 + (-1)^{n+1} \sqrt{n}} \to -1.$$

3.
i. State the Monotonic Sequence Theorem (MST).
ii. Using the Monotonic Sequence Theorem (or otherwise) prove that the sequence (a_n) defined as $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{\sqrt{(a_n)^2 + 1}} \text{ for } n > 0$$

is convergent.
iii. Prove that any sequence of real numbers has a monotonic subsequence.

4.
i. Define what it means to say that a series $\sum a_n$ is convergent.
ii. Determine whether the series $\sum a_n$ converges in each of the following cases.

(a) $a_n = \sqrt{2n+2} - \sqrt{2n}$; (d) $a_n = (-1)^n \left(\frac{1}{n} + \frac{1}{n+1}\right)$;
(b) $a_n = \frac{1+n}{1-3n^3}$; (e) $a_n = \frac{1}{\sqrt{n}} + (-1)^n \frac{1}{n}$.
(c) $a_n = \frac{2n+1}{n!}$;

[Give reasons in each case. You may use any standard tests and results without proof, provided that you make it clear which ones you are using.]

5.
i. Let (c_n) be a sequence such that the series $\sum 2^n c_n$ is convergent. Prove that the power series $\sum c_n z^n$ is absolutely convergent for any z such that $|z| < 2$.
ii. Define the radius of convergence of a power series $\sum a_n z^n$.
iii. Define the *product* of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$.
iv. What is the radius of convergence of the product of the power series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} (-1)^{n+1} z^n$?