UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2008

This paper is also taken for the relevant examination for the Associateship.

M1M2

MATHEMATICAL METHODS II

Date: Tuesday, 20th May 2008 Time: 10am -12pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Find the unique solution $y(x)$ of

$$\frac{d^2y}{dx^2} = -x\left(\frac{dy}{dx}\right)^2$$

satisfying $y\,(0) = 0$ and $\dfrac{dy}{dx}\,(1) = 1$.

(ii) Show that the differential equation

$$\left(3y^2 + 3x\right)\,\frac{dy}{dx} + (3y - 2x) = 0$$

is exact. Solve this equation.

(iii) For the function $\phi(x\,,\,y\,,\,z) = -x^2 + y^3 + 3xy - z^2$ find $\boldsymbol{u} = \text{grad}\phi \equiv \nabla\phi$ and use it to find a unit vector which is in the direction of greatest increase of ϕ at the point $\boldsymbol{r} = (-1\,,\,1\,,\,1)$.

What is $\text{div}\,\boldsymbol{u} \equiv \nabla\cdot\boldsymbol{u}$?

(iv) Evaluate the double integral $I = \int\int_R ye^{-x}dxdy$, where R is the triangular region bounded by the x-axis, the vertical line $x = 1$ and the line $y = x$.

(v) Find the position of the centre of mass of a uniform thin wire in the form of a semi-circular arc of radius a .

2. (i) Find the general solution $x(t)$ for the differential equation

$$\frac{d^2x}{dt^2} + 7\,\frac{dx}{dt} + 12\,x = 0\ .$$

(ii) Find the solution $x(t)$ for the differential equation

$$\frac{d^2x}{dt^2} + 7\,\frac{dx}{dt} + 12\,x = e^{4t} - 2e^{-3t}$$

for which $x(0) = \dfrac{1}{56}$ and $\dfrac{dx}{dt}(0) = \dfrac{1}{14}$.

(iii) By writing $y = \dfrac{dx}{dt}$ show that the differential equation in (i) above can be written in the form of a system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

with M a constant matrix. Solve this system and sketch typical trajectories in the $(x\,,y)$ phase plane.

3. (i) Consider the sum

$$S(n) = 1 + 3^3 + 5^3 + \cdots + (2n-1)^3$$

What first order difference equation does $S(n)$ satisfy?
Solve the difference equation to find an expression for $S(n)$.

(ii) Find the general solution of

$$\begin{aligned} U(n+2) - 3U(n+1) - 4U(n) \\ = 5(3)^n - (4)^n \ . \end{aligned}$$

4.

(i) Using the change of variables from Cartesians $(x\,,\,y)$ to plane polars $(r\,,\,\theta)$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial \bar{u}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \bar{u}}{\partial \theta}\right)^2$$

where $u(x\,,\,y) \equiv \bar{u}\,(r\,,\,\theta)$.

(ii) Laplace's equation in plane polars takes the form

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} = 0 \ .$$

Show that there are solutions of this equation of the form $f(r)\cos\theta$, where $f(r)$ has a specific form [to be found].

Choose arbitrary constants suitably in order to find $\bar{u}\,(r\,,\,\theta)$ which satisfies Laplace's equation, and is such that:

$$\bar{u} \to Ur\,\cos\theta \quad \text{as} \quad r \to \infty$$
$$\frac{\partial \bar{u}}{\partial r} = 0 \quad \text{on the circle} \quad r = a \ .$$

(iii) Show that the origin $(0\,,0)$ is a stationary point of the function

$$f(x\,,\,y) \;=\; xy(3x + 2y - 1)$$

and determine its character.