1. (i) Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2e^{-2x}$$

with $y(0) = 1$ and $\dfrac{dy(0)}{dx} = -5$.

(ii) Show that the differential equation

$$(2x^2 + y)\frac{dy}{dx} + (y^3 - 2xy) = 0$$

is NOT exact, but can be made exact by multiplying by a suitable power of y. Hence, find the general solution of the equation.

2. (i) Solve the difference equation

$$5U(n+2) - 4U(n+1) - U(n) = 0$$

under the conditions

$$U(1) = \kappa \quad \text{and} \quad U(n) \to 2 \quad \text{as} \quad n \to \infty$$

where κ is a constant.

Show that

$$U(n) \geq 0 \quad \text{for all} \quad n \geq 1 \text{ if and only if } 0 \leq \kappa \leq 12 \quad .$$

(ii) For the logistic map

$$x_{n+1} = ax_n(1 - x_n) \equiv F(x_n)$$

where a is constant and non-negative:

(a) Show that there is an attracting fixed point at $X_1 = 0$ for $0 \leq a \leq 1$ and at $X_2 = 1 - \frac{1}{a}$ for $1 < a < 3$, each with FIRST order convergence.

(b) For the quadratic equation with roots X_1, X_2 (as given in (a) above), construct the Newton-Raphson iterative process and show explicitly that it gives SECOND order convergence.

3. Two companies X, Y are exploring the possibilities for cooperation in their business activities. Measures of their levels of enthusiasm for this are given by $x(t), y(t)$ respectively, as functions of time t, where x, y can take positive and/or negative values.

The equations of motion are

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad .$$

(i) Show that the general solution for $x(t)$ is of the form

$$x(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t}$$

and find λ_1, λ_2 .

(ii) From the results in (i) above, find $y(t)$ and show that the trajectories in the (x, y) phase plane are given by $(2x - y)(x - 2y)^2 =$ constant.

(iii) Sketch the (x, y) phase portrait of the system. For what ranges of initial enthusiasm $x(0), y(0)$ do the enthusiasms eventually increase without limit?

4. (i) Using the change of variables

$$s = \frac{y}{x^2 + y^2} \quad , \quad t = \frac{x}{x^2 + y^2} \quad ,$$

show that

$$\left(\frac{\partial \overline{u}}{\partial s}\right)^2 + \left(\frac{\partial \overline{u}}{\partial t}\right)^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\left(x^2 + y^2\right)^2$$

where $u = u(x, y) \equiv \overline{u}(s, t)$.

(ii) A closed rectangular box has edge lengths x, y, z . If the volume V of the box is kept fixed, find the edge lengths for the rectangular box which has the smallest surface area A.

(iii) Find the Taylor expansion (up to quadratic terms) for $f(x, y) = \sin(x + 3y)$ about the point $P\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Find the direction in the (x, y) plane in which $f(x, y)$ increases most rapidly from P, and find the rate of increase in the direction $(2, 1)$.

5. Given the curve $y = x^{\frac{1}{2}}$ from $x = 0$ to $x = 1$:

(i) Find the area under the curve bounded by the x axis and the line $x = 1$.

(ii) Show that the arc length of the curve is $\frac{1}{2}\sqrt{5} + \frac{1}{4} ln\left(2 + \sqrt{5}\right)$.

(iii) Find the mass of a plane lamina cut in the shape of the region (defined in (i) above) if the density of the lamina (i.e., mass per unit area) is $\sigma = \kappa x y$, with κ constant.

(iv) Find the position of the centre of mass of the lamina defined in (iii) above.