Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2006

This paper is also taken for the relevant examination for the Associateship.

M1M2

Mathematical Methods II

Date: Monday, 15th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = -5xe^x.$$

(ii) (a) Show that the differential equation with variable coefficients for $y(x)$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

may be reduced to one with constant coefficients for $y(t)$, using the substitution $x = e^t$.

(b) Hence, or otherwise, find the general solution for $y(x)$ of the equation in (a). Find the particular solution such that

$$y(1) = 1, \quad \frac{dy}{dx}(1) = 2\,.$$

2. (i) If $S(n) = 1^2 + 4^2 + 7^2 + \;\cdots\; + (3n-2)^2$, show that

$$S(n+1) - S(n) = (3n+1)^2\,.$$

Solve this difference equation for $S(n)$.

(ii) A child who is jumping up a staircase of n steps is able to negotiate safely either one or two steps at each jump.

(a) Show that the number $U(n)$ of different ways of jumping up the staircase satisfies the difference equation

$$U(n) = U(n-1) + U(n-2)\,.$$

(b) Find the number of different ways of jumping up a staircase of 10 steps.

(c) Find an expression for the general solution $U(n)$ and use it to find

$$\lim_{n\to\infty} \frac{U(n+1)}{U(n)}\,.$$

3. The populations $x(t)$ and $y(t)$ of two species interact at time t according to the following first order linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(i) By solving the resulting eigenvalue/eigenvector problem, show that the general solution for $x(t)$ is of the form

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

with A_1, A_2, λ_1 and λ_2 constant. Hence find $y(t)$.

(ii) Solve the equation for $y(x)$:

$$\frac{dy}{dx} = \frac{2x+2y}{3x+y}$$

to show that

$$\frac{(2x+y)}{(x-y)^4} = \text{constant}.$$

(iii) Show by substitution that this solution in (ii) is in accord with that obtained in (i) above, and sketch the (x, y) phase plane portrait.

4. (i) (a) Find the stationary points of the function

$$f(x, y) = x^2y - xy^2 - 4xy$$

and determine their character.

(b) Sketch typical contours of the function $f(x, y)$, indicating the positions of the stationary points.

(ii) Given the vector differential operator

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

and that $\phi = xy^2 - 3yz$ is a scalar field,

(a) find $\mathbf{u} = \mathrm{grad}\,\phi \equiv \nabla\phi$,

(b) find a unit vector which is in the direction of greatest rate of increase of ϕ at the point $\mathbf{r} = (2, -3, 1)$,

(c) find $\mathrm{div}\,\mathbf{u}$ and $\mathrm{curl}\,\mathbf{u}$.

5. The domain R is the finite region bounded by the parabolae $y = \frac{1}{2}x^2$ and $x = 4y^2$.

(i) Express the double integral $I = \iint_R dxdy$ as a repeated integral in two ways by slicing the domain parallel to the y-axis and parallel to the x-axis respectively. [*Take care with the limits.*]

(ii) Evaluate each repeated integral, obtaining the same result that $I = \frac{1}{6}$.

(iii) If the domain R is occupied by a uniform plate with constant uniform surface density σ:

(a) determine the mass m of the plate,

(b) find the position $(\overline{x}, \overline{y})$ of the centre of mass of the plate,

(c) show that the moment of inertia (second moment) of the plate about the x-axis is $\dfrac{9m}{140}$.