1. (a) Solve

$$y'' = -xy'^2$$

subject to $y(0) = 0, y'(1) = 1$

(b) Find the general solution of

$$x^2y'' - 3xy' + 4y = x^2(\ln x + 1)$$

2. (a) Using an eigenvector method find the general solution to the coupled equations

$$\begin{aligned} \frac{dx}{dt} &= -x - 4y \\ \frac{dy}{dt} &= x - y \end{aligned}$$

and sketch the phase portrait.

(b) Also find the general solution to the inhomogenous system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} t$$

3. (a) Using a recurrence relation method sum the series

$$S(n) = \sum_{r=1}^{n} r^3$$

(b) An implicit equation for a function of two variables $z = z(x, y)$ is

$$F(x, y, z) = 0$$

Show that the partial derivative

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$$

where $F_x = (\partial F/\partial x)_{y,z}$. Hence show

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1$$

4. (a) Sketch the contours of

$$f(x, y) = xy(x + y - 2)$$

along which f is zero and indicate the regions where f is positive and negative. Locate the stationary points and deduce their nature.

(b) A function of two variables $V(x, y)$ is rewritten using new variables $s = x - y$, $t = x + y/2$. Use the chain rule to establish the operator relations.

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \\ \frac{\partial}{\partial y} &= \frac{-\partial}{\partial s} + \frac{1}{2}\frac{\partial}{\partial t}. \end{aligned}$$

Hence show that the function

$$V(x, y) = f(x - y) + g\left(x + \frac{y}{2}\right),$$

with f and g twice differentiable functions, satisfies

$$V_{xx} - V_{xy} - 2V_{yy} = 0$$

5. (a) Calculate the length of the curve

$$y = \frac{x^4}{4} + \frac{1}{8x^2}$$

between $x = 1$ and $x = 2$

(b) A uniform circular wire of radius R has a section of arc length $2l$ cut from it.

Show that the position of the centre of mass of the remaining piece of wire is at $\bar{y} = 0$ and

$$\bar{x} = \frac{R \sin l/R}{\pi - l/R}$$