Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2007

M1M1 (Test)

Mathematical Methods 1

- Affix ONE label to each answer book that you use. DO NOT use the label with your name on it.
- Write your answers in a single answer book, using continuation books if necessary.

- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (a) What is the maximal domain of the real, (single-valued) function

$$f(x) = \sin^{-1}(e^x - 1)?$$

Find the inverse function $f^{-1}(x)$ and express the even part of $f^{-1}(x)$ in as simple a form as possible.

(b) Explain what is meant by the statement: "The function $f(x)$ is differentiable at $x = a$."
Calculate from first principles the derivative of the function

$$f(x) = \frac{1}{1+x}.$$

(c) Sketch the curve

$$y^2 = \frac{4-x^2}{1-x^2}.$$

You should identify any stationary points, but need not locate any points of inflection.

(d) Using any method, evaluate the limits

$$(i) \qquad \lim_{x \to 2} \left(\frac{\sin^2 \pi x}{x^3 - 5x^2 + 8x - 4} \right)$$

$$(ii) \qquad \lim_{x \to \infty} \left(\frac{\sin x}{x} + \left(\frac{x+3}{x-1} \right)^x \right)$$

(e) Find all complex solutions to the equations

$$(i) \qquad e^z = -1 + i \qquad\qquad (ii) \qquad 2\mathrm{Re}(z)\ \mathrm{Im}(z) = |z|^2$$

(f) Evaluate the definite integrals

$$(i) \qquad \int_0^1 \frac{\log(\tan^{-1} x)}{1+x^2}\, dx \qquad\qquad (ii) \qquad \int_0^1 \frac{x+2}{x^2+2x+2}\, dx$$

SECTION B

2. Show that if $y = \sinh^{-1} x$ then

$$y' = \frac{1}{\sqrt{1+x^2}} \qquad \text{and} \qquad (1+x^2)y'' + xy' = 0$$

Differentiating this equation n times, show that for $n \geqslant 0$

$$y^{(n+2)}(0) = -n^2 y^{(n)}(0)$$

and deduce that the Maclaurin series for y gives

$$\sinh^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{(1)^2(3)^2(5)^2 \ldots (2k-1)^2}{(2k+1)!} \, x^{2k+1}.$$

What is the radius of convergence of this series?

3. The function $y(x)$ satisifies the linear ordinary differential equation

$$\frac{dy}{dx} + \frac{4y \sin x}{5 + 4\cos x} = \frac{3}{2}$$

Find the general solution of this equation.

If in addition it is known that $y(0) = 0$, show that

$$y = (5 + 4\cos x) \tan^{-1}\left(\tfrac{1}{3} \tan(x/2)\right).$$