Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2006

M1M1 (Test)

Mathematical Methods I

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (i) Define the function $y(x)$ to be

$$y(x) = \frac{2x^2 - 1}{x + 1} \ .$$

(a) Decompose this function into the sum of an even function and an odd function.

(b) Sketch a graph of $y(x)$ carefully indicating on your sketch any important features.

(ii) Find the derivative of the function $\sin(1 + x^2)$ from first principles.

(iii) (a) Find the value of the limit

$$\lim_{x \to \infty} \left(\frac{2 \sinh x + \cosh x}{3 \sinh x - \cosh x} \right) .$$

(b) Find the value of the limit

$$\lim_{x \to 0} (\cos x)^{1/x}.$$

Calculate the value of the definite integral

$$\int_0^1 \frac{e^x}{e^{2x} + 1} \, dx \ .$$

(iv) (a) Find all complex roots of the equation

$$z^6 + 2z^4 + 2z^2 + 1 = 0.$$

(b) Find all complex roots of the equation

$$\tanh z = 2.$$

(v) Find the solution of the differential equation

$$(x + y^2)\frac{dy}{dx} + 1 = 0$$

satisfying the condition $y(1) = 0$.

SECTION B

2. Define the function $y(x)$ to be

$$y(x) = \sin^{-1}(x)$$

where it is specified that $y(0) = 0$.

(a) Find the first three non-zero terms in the Taylor series expansion of $y(x)$ about $x = 0$.

(b) Show that $y(x)$ satisfies the second order differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0 \ .$$

(c) By making use of the Leibniz rule, or otherwise, use the differential equation just derived to show that

$$y^{(n+2)}(0) = n^2 y^{(n)}(0)$$

where $y^{(k)}$ denotes the k-th derivative of $y(x)$ with respect to x.

(d) Hence show that the complete series expansion of $y(x)$ about $x = 0$ is

$$y(x) = \sum_{n=0}^{\infty} a_n x^{2n+1}$$

where

$$a_n = \frac{1}{(2n+1)!}\left(\frac{(2n)!}{2^n n!}\right)^2 .$$

(e) Verify that the values of a_0, a_1 and a_2 agree with the values you obtained in part (a).

3. (a) Find the indefinite integral $$\int x \log x \ dx \ .$$

(b) Find the indefinite integral $$\int x \sin^{-1} x \ dx \ .$$

(c) Define $$I_n = \int_0^{\infty} x^{3n} e^{-x^3/3} \ dx \ .$$

Show that, for $n \geq 1$,

$$I_n = (3n - 2) I_{n-1} \ .$$

Hence show that $I_5 = 3640 \ C$ where

$$C = \int_0^{\infty} e^{-x^3/3} \ dx \ .$$