Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The function $f(x)$ is defined for positive values of x by

$$f(x) = x^{1/x}.$$

(a) Write $f(x) = \exp[g(x)]$ for a suitable function $g(x)$.

(b) Find the limits of $f(x)$ as $x \to 0$ and as $x \to \infty$.

(c) Find the maximum value attained by $f(x)$ for $x > 0$.

(d) Sketch the curve $y = f(x)$ for $x > 0$.

(e) Find the first three terms in the Taylor series of $f(x)$ about $x = 1$.

(f) Show that the negative values of x for which $f(x)$ takes a real value are of the form $(2k+1)/n$ where k and n are integers.

(g) Determine whether or not the integral

$$\int_0^\infty \frac{(x+1)\sin x}{x^{3/2}(x-\pi)}\, dx$$

exists (do NOT try to evaluate it.)

(h) Solve the differential equation

$$e^x \frac{dy}{dx} = \sinh x \cos^2 y.$$

2. For some positive integer n, x and y are defined in terms of a parameter θ by

$$x = \cos\theta, \qquad y = \cos(n\theta).$$

(a) Using De Moivre's Theorem and the Binomial Theorem, show that

$$y = \sum_{m=0}^{p} \binom{n}{2m} x^{n-2m}(x^2-1)^m,$$

where p is the largest integer with $2p \leqslant n$ and $\binom{k}{l}$ denotes a binomial coefficient.

(b) Show that

$$\sin\theta \frac{dy}{dx} = n\sin n\theta$$

and hence that $y(x)$ obeys

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

Using Leibniz' formula, deduce that

$$y^{(k+2)}(0) = (k^2 - n^2)y^{(k)}(0) \qquad \text{for} \quad k \geqslant 0,$$

where $y^{(k)}(0)$ denotes the k'th derivative of y with respect to x evaluated at $x = 0$.
If n is a multiple of 4 show that $y(0) = 1$ and $y'(0) = 0$. Hence find the Maclaurin series for the function $y(x)$ when n is a multiple of 4, including terms up to x^4.
Verify that parts (a) and (b) give the same answer when $n = 4$.

3. (a) Rolle's theorem states that if $h(x)$ is a continuous function, is differentiable in $a < x < b$ and if $h(a) = h(b)$, then there exists a value ξ such that $a < \xi < b$ and $h'(\xi) = 0$.

Given two differentiable functions $f(x)$ and $g(x)$, consider the function

$$h(x) = f(x) + mg(x)$$

where m is a suitably chosen constant. Use Rolle's theorem to prove that provided $g(a) \neq g(b)$ there exists a ξ in $a < \xi < b$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Deduce that if $f(c) = 0 = g(c)$, then

$$\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to c} \left[\frac{f'(x)}{g'(x)} \right],$$

assuming the latter limit exists.

(b) Using any method, find the limit

$$\lim_{x \to 0} \left[\frac{1 - \cos x - \sin x + \log(1 + x)}{(1 + x^3)^{1/2} - 1} \right].$$

(c) The functions $f(x)$, $f_e(x)$ and $f_o(x)$ are defined by

$$f(x) = \frac{1}{1 - x} = f_e(x) + f_o(x),$$

where f_e is an even function and f_o is an odd function. Sketch the three functions f, f_e and f_o.

4. The function $y(x)$ obeys the **second order** linear ODE, which you have not met before,

$$y'' + a(x)y' + b(x)y = c(x),$$

where $a(x)$, $b(x)$ and $c(x)$ are given functions.

Suppose that $y = f(x)$ is a solution of this equation when $c(x) = 0$. Use the substitution $y = f(x)u(x)$ to obtain an equation for $u(x)$ in terms of a, b, c, and f. What kind of ODE do you obtain if you write $v(x) = u'(x)$? Explain how to obtain the general solution for $y(x)$ in terms of integrals of known functions.

Show that $y = x$ is a particular solution to the problem

$$x^2y'' - (x^2 + 2x)y' + (x + 2)y = 0.$$

Hence find the general solution.