Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods I

Date: Wednesday, 10th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) The function $f(x)$ is defined as

$$f(x) = \cosh(x + x^2).$$

(i) Write $f(x)$ as the sum of an even function and an odd function.

(ii) Find the first three non-zero terms in the series expansion of $f(x)$ about $x = 0$.

(iii) Find the derivative of $f(x)$ from first principles.

(b) Sketch the curve defined by the relation

$$y^2 = x^3(1 - x^3)$$

carefully indicating any important features on your sketch.

2. (a) For any integer $n \geq 0$, define the integrals

$$I_n = \int_0^\infty e^{-x} \cos nx \; dx \,,$$
$$J_n = \int_0^\infty e^{-x} \sin nx \; dx \,.$$

Show that

$$I_n + iJ_n = \frac{1}{1 - in} \,.$$

Hence, or otherwise, find I_n and J_n as functions of n.

(b) Find the indefinite integral

$$\int \frac{d\theta}{1 + \cos\theta} \,.$$

(c) Find the indefinite integral

$$\int \frac{d\theta}{1 + \cos^2\theta} \,.$$

3. Define

$$f(x) = \tanh^{-1}(x)\,.$$

(a) Find an expression for $f(x)$ in terms of the logarithm function.

(b) Hence, or otherwise, show that the n-th derivative of $f(x)$, for $n \geq 1$, is given by

$$\frac{d^n f}{dx^n} = \frac{(n-1)!}{2}\left(\frac{(-1)^{n-1}}{(1+x)^n} + \frac{1}{(1-x)^n}\right)\,.$$

(c) Find the complete Taylor series of $f(x)$ about $x = 0$.

(d) Let the function $F(x)$ be defined by

$$F(x) = \int_0^x f(x)\; dx\,.$$

By using integration by parts to find $F(x)$ explicitly, show that

$$F(1/2) = \log\left(\frac{3^{3/4}}{2}\right)\,.$$

4. (a) Find all complex roots of the equation

$$\cosh z + 2\,\sinh z = 1\,.$$

(b) Sketch all points z in the complex plane satisfying the equation

$$\left|\frac{z-i}{z+i}\right| = c\,,$$

where

(i) c=1 ;

(ii) c=2 .

(c) Show that if the complex variables ζ and z are related via the relation

$$z = \frac{2}{\zeta} + \zeta\,,$$

then the unit circle $|\zeta| = 1$ in the ζ-plane maps to an ellipse in the z-plane.

If $z = x + iy$, find the equation for this ellipse in terms of x and y.

5. (a) Find the general solution of the equation

$$\frac{d^2T}{dr^2} - \frac{2}{r}\frac{dT}{dr} = r^2.$$

(b) Find the general solution of the equation

$$\left(\frac{x+y}{x-y}\right)\frac{dy}{dx} = 1\,.$$

(c) Find the general solution of the equation

$$\frac{dy}{dx} = \sec x \sec y\,.$$