M1M1 Mathematical Methods
Summer Examination 2004

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1. Consider the function
$$f(x) = \frac{x^3 - 1}{x^3 + 1}.$$

(i) Put $f(x)$ in partial fraction form;

(ii) Find and classify all the stationary points of $f(x)$;

(iii) Find all the points of inflexion;

(iv) Sketch the graph of $f(x)$, carefully indicating all the important features on your sketch (including the stationary points and points of inflexion);

(v) Write $f(x)$ as the sum of an odd function of x and an even function of x.

2. If
$$\zeta = \cot\left(\frac{\theta}{2}\right) e^{i\phi}$$
where θ and ϕ are real, show that
$$\sin\theta = \frac{2\sqrt{\zeta\bar{\zeta}}}{1 + \zeta\bar{\zeta}},$$
$$\cos\theta = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}.$$

Let z be some complex number. Show that the transformation
$$z \mapsto \frac{z+2}{z-2}$$
maps the unit circle in the z-plane to another circle and find the centre and radius of this circle.

Sketch the locus of the curve given by $|z - 1| = 2|z + 1|$.

3. Find the following integrals:

(a)
$$\int \frac{dx}{x^3 - x^2 + 2x - 2};$$

(b)
$$\int \frac{dx}{e^{2x} + 1};$$

(c)
$$\int \frac{dx}{\cos x - \sqrt{2}\sin x};$$

(d)
$$\int \tanh^{-1} x \; dx.$$

Let
$$I_n = \int_0^{\pi/4} \sec^n x \; dx.$$
Determine a recurrence relation relating I_n and I_{n-2} for $n \geq 2$.

4. Consider the function $\tanh^{-1} x$.

(a) Find an expression for $\tanh^{-1} x$ in terms of logarithms;

(b) Find an expression for the n-th derivative of $\tanh^{-1} x$ where $n \geq 1$ is any integer;

(c) Find the complete Taylor expansion of $\tanh^{-1} x$ about $x = 0$;

(d) Suppose the approximation
$$\tanh^{-1} x \approx x$$
is used to approximate the function $\tanh^{-1} x$ in the interval $0 \leq x \leq 1/2$, use Taylor's theorem to find an estimate of the maximum possible error incurred in using this approximation.

5. Find the general solutions of the following ordinary differential equations:

(a)

$$\frac{dy}{dt} - \frac{ty}{t^2+1} = 1;$$

(b)

$$\frac{dy}{dt} = -1 - \frac{2t}{y-t}.$$

Find the solution of the coupled system of ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} + yx &= 1, \\ \frac{dy}{dt} + \frac{y}{t} &= 0 \end{aligned}$$

with $x(1) = 1, y(1) = 2$.