M1M1 Mathematical Methods
Summer Examination 2003

1.

(a) Consider the fourth-order polynomial function with real coefficients given by $P(z) = z^4 + 1$.

(i) Find all the complex roots of $P(z)$.

(ii) $P(z)$ can be factored into two quadratic polynomials with real coefficients. Using part (i) or otherwise, find these two quadratic factors.

(b) Find all complex solutions to the equation

$$\sinh z = 1.$$

2.

(a) Find the full Taylor expansion, about $x = 0$, of the function $\log(1+x)$. Use the ratio test to find its radius of convergence.

(b) Let $f(x) = (1+x)^2 \log(1+x)$. Find the first 3 non-zero terms in the Taylor expansion of $f(x)$ about $x = 0$.

(c) Using the Leibniz rule, find an expression for

$$\frac{d^n f(x)}{dx^n} \quad \text{for} \quad n \geq 3.$$

(d) Use your answer to part (c) to check one of the coefficients obtained in part (b), explaining your check carefully.

3.

Find the solutions to the following ordinary differential equations:

(a)

$$\frac{dy}{dx} = \frac{x-y}{x-2y}; \qquad y(1) = 0.$$

(b)

$$\frac{dy}{dx} = \frac{\log x}{y^2}; \qquad y(1) = 0.$$

(c)

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = e^{-x^2/2}; \qquad y(0) = 0,\ y'(0) = 0.$$

4.

(a) Find the derivatives of

$$e^{\sin x}, \quad \sin(e^x), \quad (\sin x)^x.$$

(b) Consider the curve defined by

$$y^2 = x + x^3.$$

(i) Does the curve have any stationary points? If so, find them.

(ii) Find any points at which the tangent to the curve becomes vertical (i.e. points where the tangent has infinite gradient).

(iii) Find any points of inflexion.

(iv) Sketch the curve, carefully indicating any stationary point, points of infinite gradient and inflexion points on your sketch.

5.

(a) Find the following indefinite integrals:

$$\int \frac{e^x + 1}{e^x + 2}dx; \qquad \int \frac{d\theta}{\sin\theta + \cos\theta}.$$

(b) Let $n \geq 0$ be an integer. Define

$$I_n = \int_0^1 [\log(x)]^n dx.$$

Compute I_{100}.