M1M1 Mathematical Methods
Summer Examination 2002

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1. Consider the function $f(x)$ defined as

$$f(x) = \frac{x^2+3}{x-1}.$$

(a) By writing the function in the form

$$f(x) = Ax + B + \frac{C}{x-1}$$

where A, B and C are constants to be determined, find a general formula for its n-th derivative, i.e.,

$$\frac{d^n f(x)}{dx^n}$$

where $n \geq 2$.

(b) Sketch a graph of the function $f(x)$, carefully labelling any stationary points and asymptotes on your graph.

(c) Determine the inverse function $f^{-1}(x)$ for $x \geq 6$.

(d) Express $f(x)$ as the sum of an even and an odd function.

2. Solve the following differential equations:

(a)

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^3} \qquad \text{with } y(1) = 0;$$

(b)

$$\frac{dy}{dx} + \frac{3y}{x} = \left(\frac{y}{x}\right)^2 \qquad \text{with } y(1) = 1;$$

(c)

$$\frac{dy}{dx} = \frac{1}{x+y} \qquad \text{with } y(1) = 0.$$

Hint for part (c): consider the equation for $\frac{dx}{dy}$.

3. (a) Find all complex solutions z of the equation

$$\sin z = -2.$$

(b) Find all complex solutions z of the equation

$$z^3\bar{z} - z^2 + 4z\bar{z} - 4 = 0$$

and sketch the solutions in the complex plane.

(c) It is known that $\cos 4\theta$ can be written as a polynomial in $\cos\theta$, i.e.

$$\cos 4\theta = c_4 \cos^4\theta + c_3 \cos^3\theta + c_2 \cos^2\theta + c_1 \cos\theta + c_0.$$

Use De Moivre's theorem to determine the five coefficients c_j $(j = 0, 1, 2, 3, 4)$.

4. Consider the function

$$f(x) = \frac{e^{-x}}{1-x}.$$

(a) By multiplying together the well-known Taylor series expansions of e^{-x} and $\frac{1}{1-x}$, find the first 4 non-zero terms in the Taylor series expansion of $f(x)$ about $x = 0$.

(b) Find a general formula for

$$\frac{d^n f(x)}{dx^n}$$

where $n \geq 0$ is an integer.

(c) Denote the Taylor series expansion of $f(x)$ about $x = 0$ as follows

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using your result in part (b), find a general formula for a_n for any $n \geq 0$.

(d) Check your answer to part (c) using your answer in part (a).

5.

(a) For any integer $n \geq 0$, define the quantity I_n as follows:

$$I_n = \int_0^{\pi} \cos^n x dx.$$

(i) Derive a recurrence formula relating I_n and I_{n-2} for $n \geq 2$.

(ii) Find the value of I_{10}.

(b) Evaluate the indefinite integrals

$$\int \sec x \; dx; \qquad \int \sec^3 x \; dx.$$