BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2006

M1GLA (Test)

Geometry and Linear Algebra

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

It is enough just to write down your answers to each part of this question. A correct answer will score full marks, and an incorrect one will score no marks.

1. (i) Find, in the form $px_1 + qx_2 + r = 0$, the equation of a line in $\mathbb{R}^2$ having normal $(-1, 1)$ and at a perpendicular distance of $\sqrt{2}$ from the point $(3, 1)$.

(ii) True or false: $||x - y|| \geq ||x|| - ||y||$ for all vectors $x, y \in \mathbb{R}^2$?

(iii) Find values of α and β such that the following system of linear equations has an infinite number of solutions:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & x_2 & + & \alpha x_3 & = & -1 \\ 3x_1 & + & x_2 & + & x_3 & = & \beta \end{array}$$

(iv) For which values of α and β does the system in the previous part have no solutions ?

(v) Let $A = \begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}$. Find the eigenvalues of A.

(vi) For the matrix A in part (v), find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.

(vii) For the matrix A in part (v), find A^7.

(viii) Find positive values of r, s, t such that the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & r \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & s \\ \frac{1}{\sqrt{3}} & 0 & -t \end{pmatrix}$$

is orthogonal.

(ix) Let A, B, C be the points $(0, -1, 2)$, $(2, -1, 1)$, $(-2, -3, 1)$ in $\mathbb{R}^3$. Find the area of the triangle ABC.

(x) Find a unit vector normal to the plane containing the points A, B, C of the previous part.

SECTION B

2. Let α be a real number and let A be the matrix $\begin{pmatrix} 1 & \alpha \\ \alpha & 4 \end{pmatrix}$.

   (a) Find the eigenvalues of A (in terms of α).

   (b) Find a positive value of α such that one of the eigenvalues of A is 0.

   (c) For the value of α found in part (b), find a 2×2 rotation matrix P such that $P^T AP$ is diagonal.

   (d) For the value of α found in part (b), show that the conic

$$x_1^2 + 2\alpha x_1 x_2 + 4x_2^2 - \sqrt{5}(2x_1 - x_2) = 0$$

   is a parabola, and make a rough sketch of it in the x_1x_2-plane.

3. Let e_1, e_2, e_3 be the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ in $\mathbb{R}^3$.

   (a) Find all vectors $x \in \mathbb{R}^3$ such that

$$x + (x.e_1)\, e_2 = e_3.$$

   (b) Find all vectors $x \in \mathbb{R}^3$ such that

$$x + (x \times e_1) = e_2.$$

   (c) Suppose that x, y are vectors in $\mathbb{R}^3$ such that

$$x + (x \times y) = y.$$

   Prove that $x = y$.

   (You may use any standard properties of the vector product that you require.)