Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M1GLA

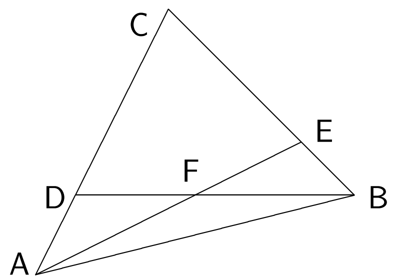
Geometry and Linear Algebra

Date: Friday 12th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let A, B, C be three points in $\mathbb{R}^2$, not all on the same line. Let D be the point on AC such that $AD = \frac{1}{3}AC$, and let E be the point on BC such that $BE = \frac{1}{3}BC$. Let F be the point of intersection of the lines AE and BD. Here is the picture:



Let a, b, c, d, e, f be the position vectors of the points A, B, C, D, E, F, respectively.

(i) Find expressions for d and e in terms of a, b, c.

(ii) Find f in terms of a, b, c.

(iii) Into what ratio does F divide the lines AE and BD?

(iv) Prove that the line CF bisects AB.

2. (a) Let A be an $m \times n$ matrix, and let b be a column vector in $\mathbb{R}^m$. Define two systems of linear equations as follows:

(1) $Ax = b$

(2) $Ax = 0$

where $x \in \mathbb{R}^n$.

Let p be a solution of the system (1) (i.e. $p \in \mathbb{R}^n$ and $Ap = b$). Prove that every solution of (1) is of the form $p + h$, where h is a solution of the system (2).

(b) Find all solutions of the system

$$\begin{aligned} x_1 + x_3 + x_5 &= 1 \\ x_2 + x_4 + x_5 &= 0 \\ x_1 + x_4 + x_5 &= 2 \end{aligned}$$

Express the general solution in the form $p + h$, as in part (a).

Question 2 is continued on Page 3

2. (c) Consider the system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 6x_2 & + & \alpha x_3 & = & \alpha+3 \\ x_1 & + & x_2 & + & (1-\alpha)x_3 & = & 2-\alpha \end{array}$$

where α is a real number.

(i) Find a value of α for which this system has no solutions.

(ii) How many solutions does the system have when α is not equal to the value in part (i)?

3. Recall that the conic in $\mathbb{R}^2$ with focus $p = (p_1, p_2)$, directrix the line L, and eccentricity $e \in \mathbb{R}$, is defined to be the curve consisting of points $x = (x_1, x_2)$ satisfying the equation

$$d(x, p) = e\, d(x, L),$$

where $d(x, p)$ is the distance between x and p, and $d(x, L)$ is the perpendicular distance from x to L.

(a) Find the equation (in terms of the coordinates x_1, x_2) of the conic with focus $(1, 2)$, directrix $x_1 + 2x_2 = 1$, and eccentricity $e = \sqrt{5}$.

(b) Make a rough sketch in the x_1x_2-plane of the conic in part (a). What kind of conic is it?

(c) Find a rotation matrix P such that the change of coordinates $x = Py$ reduces the equation of the conic in part (a) to standard form.

4. (a) Suppose that A and P are $n \times n$ matrices, with P invertible, such that $P^{-1}AP$ is a diagonal matrix. Prove that each column of P is an eigenvector of A.

(b) Let $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$. Prove that there is no invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.

(c) Let $A = \begin{pmatrix} -7 & 6 \\ -9 & 8 \end{pmatrix}$. Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.

(d) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Does there exist an invertible 3×3 matrix P such that $P^{-1}AP$ is diagonal? Give your reasoning.

5. (a) Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors in $\mathbb{R}^3$. Write down the coordinates of the vector $x \times y$.

(b) Prove that $x \times y$ is perpendicular to both x and y.

(c) Let α be a real number, let L_1 be the line in $\mathbb{R}^3$ containing the points $(1, 1, 0)$ and $(0, -1, \alpha)$, and let L_2 be the line containing the points $(1, 0, -1)$ and $(-1, 1, \alpha + 1)$.

(d) (i) Find (in terms of α) a vector which is perpendicular to both the lines L_1 and L_2.

(ii) Given that there is a plane which contains both L_1 and L_2, find the value of α.

(iii) For the value of α in part (ii), find the point of intersection of L_1 and L_2.