1. Define the length $||x||$ of a vector $x \in \mathbb{R}^2$.

Which of the following statements are true and which are false? For those which are false, give a counterexample. For those which are true give a brief proof (you may use any standard results you require without proof, provided you state them clearly).

(i) $||x + 2y|| \leq ||x|| + 2||y||$ for all $x, y, \in \mathbb{R}^2$.

(ii) $||x + 2y|| \geq 2||y|| - ||x||$ for all $x, y \in \mathbb{R}^2$.

(iii) $||x + 2y|| \geq ||x|| - ||y||$ for all $x, y \in \mathbb{R}^2$.

(iv) If A is a 2×2 matrix whose columns are both unit vectors, then $||Ax|| = ||x||$ for all $x \in \mathbb{R}^2$.

(v) If A is a 2×2 matrix whose rows are unit vectors which are perpendicular to each other, then $||Ax|| = ||x||$ for all $x \in \mathbb{R}^2$.

2. Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & a & 0 \\ 1 & 1 & -a \end{pmatrix}$.

(i) Find the determinant $|B|$ (in terms of a).

(ii) For which values of a is B invertible?

(iii) For which values of a and b does the system

$$\begin{array}{lclclcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & ax_2 & & & = & 0 \\ x_1 & + & x_2 & - & ax_3 & = & b \end{array}$$

have

(a) no solutions

(b) exactly one solution

(c) infinitely many solutions?

In case (c), find the general solution to the system.

3. Let $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find the eigenvalues and eigenvectors of A.

(ii) Let $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

By expressing v and w in terms of eigenvectors of A (or otherwise), calculate the vectors

$$A^{11}v \quad \text{and} \quad A^{11}w.$$

(iii) Hence solve for $x \in \mathbb{R}^3$ the system

$$A^{11}x = \begin{pmatrix} -5 \\ 5 \\ 2 \end{pmatrix}.$$

4. (i) Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in \mathbb{R}^3$.

Write down the coordinates of the vector product $a \times b$.

Prove that $a \times b$ is perpendicular to both a and b.

(ii) Let $x \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ be such that

$$x \times (1, 0, 1) = (-1, 2, \lambda).$$

Find the value of λ.

Find x, given that $||x|| = \sqrt{3}$.

Prove that there are no solutions for x with $||x|| = \sqrt{2}$.

5. (i) What is meant by an orthogonal matrix?

(ii) Find positive real numbers a, b, c, d such that the matrix

$$P = \begin{pmatrix} -a & b & d \\ 0 & -c & c \\ a & b & d \end{pmatrix}$$

is an orthogonal matrix of determinant 1.

(iii) Let P be the orthogonal matrix in part (ii), and let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Show that the matrix $P^\top AP$ is diagonal.

(iv) Reduce the equation of the quadric surface

$$2x_1x_2 + 2x_2x_3 = 1$$

to standard form, and determine the nature of this quadric surface.