1. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false.

(i) $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in \mathbb{R}^2$.

(ii) $||x-y|| \geq ||y|| - ||x||$ for all $x, y \in \mathbb{R}^2$.

(iii) If $x, y \in \mathbb{R}^2$ and $||x-y|| = ||y|| - ||x||$, then $y = \lambda x$ for some $\lambda \in \mathbb{R}$.

(iv) If three points in the plane with position vectors $x, y, z \in \mathbb{R}^2$ all lie on the same straight line L, then the point with position vector $\frac{1}{3}(x+y+z)$ also lies on L.

2. (a) Prove that if v is a column vector in $\mathbb{R}^2$, and $v \neq 0$, then there exists an invertible 2×2 matrix A such that

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v.$$

(b) Prove that if v and w are column vectors in $\mathbb{R}^2$ with $v, w \neq 0$, then there exists a 2×2 matrix A such that

$$Av = w.$$

(c) Let v_1, v_2, w_1, w_2 be non-zero column vectors in $\mathbb{R}^2$, such that v_1 is not a scalar multiple of v_2, and w_1 is not a scalar multiple of w_2. Prove that there exists a 2×2 matrix A such that

$$Av_1 = w_1 \text{ and } Av_2 = w_2.$$

3. (a) Let A be a 3×3 matrix and b a 3×1 column vector. Prove the following statements concerning the system of linear equations $Ax = b$ (where x is a 3×1 column vector of unknowns). You may use any standard results you require provided you state them clearly.

(i) If $\det(A) \neq 0$ then the system $Ax = b$ has exactly one solution.

(ii) If $\det(A) = 0$ then the system $Ax = b$ has either no solutions or infinitely many solutions.

(b) For which values of the real number α does the system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & - & 2x_2 & + & 4x_3 & = & 1 \\ x_1 & + & \alpha x_2 & + & \alpha^2 x_3 & = & \alpha^3 \end{array}$$

have

(i) no solutions ?

(ii) exactly one solution ?

(iii) infinitely many solutions ?

4. Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

(i) You are given that 5 is an eigenvalue of A. Find the other eigenvalues.

(ii) Find the eigenvectors of A.

(iii) Find an orthogonal 3×3 matrix P (i.e. a matrix satisfying $P^T P = I$) such that $P^T AP$ is a diagonal matrix.

(iv) Giving your reasoning, say what kind of quadric surface in $\mathbb{R}^3$ is represented by the equation

$$3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3 = 0.$$

5. (a) Define the vector product $a \times b$ of two vectors $a, b \in \mathbb{R}^3$.

(b) Let A, B, C be the points $(0, 1, 1)$, $(1, 0, -1)$, $(2, 1, 0)$ respectively.

(i) Find the area of the triangle ABC.

(ii) Find the volume of the parallelepiped with corners A, B, C, D, where D is $(-2, 0, 2)$.

(iii) Find α such that the four points A, B, C and $(1, 2, \alpha)$ are coplanar (i.e. lie on the same plane).

(c) Let $a \in \mathbb{R}^3$ be a non-zero vector, and suppose that $x, y \in \mathbb{R}^3$ satisfy the equations

$$a.x = a.y \quad \text{and} \quad a \times x = a \times y.$$

Prove that $x = y$.

(You may assume the identity $(b \times c) \times d = (b.d)c - (c.d)b$ for any $b, c, d \in \mathbb{R}^3$.)