M1GLA Exam, May 2003

1. Let A, B be two distinct points in the plane with position vectors $a, b \in \mathbb{R}^2$, respectively.

Write down in vector form a straight line L which passes through A and B.

Prove that L is the only line passing through A and B.

Prove that the equation of the perpendicular bisector of the line segment AB is

$$x.(a - b) = \frac{1}{2}(||a||^2 - ||b||^2).$$

2. In this question, A is an $m \times n$ matrix, b an $m \times 1$ column vector, and x an $n \times 1$ column vector of unknowns.

(i) Prove that the system of linear equations $Ax = b$ either has no solutions, or has exactly 1 solution, or has infinitely many solutions for x.

(ii) Show that if the system $Ax = b$ has exactly 1 solution, then the system $Ax = 0$ also has exactly 1 solution.

(iii) Show that if the system $Ax = 0$ has exactly 1 solution, then the number of solutions of the system $Ax = b$ is either 0 or 1.

(iv) For which values of λ does the system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & \lambda \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \\ x_1 & - & 3x_2 & + & x_3 & + & \lambda x_4 & = & 2 \end{array}$$

have no solutions ?

3. Let $A = \begin{pmatrix} -6 & 5 \\ -10 & 9 \end{pmatrix}$.

(i) Find the eigenvalues and eigenvectors of A.

(ii) Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.

(iii) Find a 2×2 matrix C with complex entries such that $C^2 = A$.

(iv) Show that if X is an arbitrary 2×2 matrix, then $\det(X^2) = (\det X)^2$.

(v) Using (iv), prove that there is no 2×2 matrix D with real entries such that $D^2 = A$.

4. Let $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, a symmetric 2×2 matrix with real entries a, b, c, where $b \neq 0$.

(i) Prove that S has two different real eigenvalues α_1, α_2.

(ii) Let v_1, v_2 be eigenvectors of S corresponding to the eigenvalues α_1, α_2 respectively. Prove that v_1 is perpendicular to v_2 (i.e. $v_1^T v_2 = 0$).

(iii) Now let $S = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$. Find a 2×2 rotation matrix P such that $P^T SP$ is diagonal.

(iv) Giving your reasoning, say what type of conic is defined by the equation

$$5x_1^2 + 4x_1x_2 + 2x_2^2 = 1.$$

5. Define the vector product $a \times b$ of two vectors $a, b \in \mathbb{R}^3$.

Show that $a \times b$ is perpendicular to both a and b.

Now let A, B, C be three points, with position vectors $a, b, c \in \mathbb{R}^3$, and suppose A, B, C are not collinear (i.e. do not all lie on the same straight line). Find, in terms of a, b, c, a non-zero vector which is normal to the plane containing A, B, C, and show that the equation of this plane is

$$x.((a \times b) + (b \times c) + (c \times a)) = a.(b \times c).$$

Deduce that the point with position vector $a + b + c$ lies in this plane if and only if $a.(b \times c) = 0$.