M1GLA Exam 2002

1. Let x, y be vectors in $\mathbb{R}^2$.

(i) Define the scalar product $x.y$ and the length $||x||$.

(ii) Prove that

$$||x+y||^2 = ||x||^2 + ||y||^2 + 2x.y$$

(iii) Prove the Cauchy-Schwarz inequality

$$x.y \leq ||x||\,||y||.$$

(You may not assume any trigonometry results, such as the cosine rule, without proof.)

(iv) Deduce the triangle inequality

$$||x+y|| \leq ||x|| + ||y||.$$

(v) Show that if $||x+y|| = ||x|| + ||y||$, and $x \neq 0$, then y is a scalar multiple of x.

2. Find in terms of c the eigenvalues of the matrix

$$\begin{pmatrix} 1 & c \\ c & 3 \end{pmatrix}$$

Find a positive value of c such that one of these eigenvalues is 0.

For this value of c, consider the conic

$$x_1^2 + 2cx_1x_2 + 3x_2^2 = \sqrt{3}x_1 - x_2$$

Find a rotation matrix P such that the change of variables $x = Py$ reduces the equation to standard form. Show that the conic is a parabola and sketch it in the x_1x_2-plane.

3. Let A be an $m \times n$ matrix and b an $m \times 1$ column vector. Explain why if $m < n$, then the system of linear equations

$$Ax = b$$

has either no solutions, or has infinitely many solutions for x, where x is an $n \times 1$ column vector of unknowns. (Just an explanation is required, no proofs necessary.)

Consider the system of linear equations

$$\begin{array}{lllllllll} x_1 & + & x_2 & + & x_3 & + & x_4 & = & -1 \\ 2x_1 & + & ax_2 & + & x_3 & + & x_4 & = & 1 \\ 3x_1 & + & x_2 & + & x_3 & + & ax_4 & = & b \end{array}$$

where a, b are real numbers. For which values of a and b does the system have

(i) no solutions ?

(ii) infinitely many solutions ?

4. Let A be the matrix $\begin{pmatrix} -1 & -2 & 1 \\ 0 & -3 & 1 \\ 0 & -6 & 2 \end{pmatrix}$.

(i) Find the eigenvalues of A.

(ii) Find an invertible 3×3 matrix P such that $P^{-1}AP$ is diagonal.

(iii) Show that $A^n = (-1)^{n-1}A$ for all integers $n \geq 2$.

5. Let a, b, c be vectors in $\mathbb{R}^3$.

(i) Define the vector product $a \times b$.

(ii) Show that $(a \times b) \times c = (a.c)b - (b.c)a$.

(iii) Deduce the Jacobi identity

$$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0.$$

(iv) Find a real number α such that the four points

$$(1, 0, 0),\ (2, 3, -4),\ (1, \alpha, -\alpha),\ (0, -2\alpha, \alpha)$$

are distinct and coplanar.