Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2007

M1F (Test)

Foundations of Analysis

- Affix ONE label to each answer book that you use. DO NOT use the label with your name on it.
- Write your answers in a single answer book, using continuation books if necessary.

- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (i) Let S_1 be the union of the irrational numbers with the integers.
Let S_2 be the set of all real numbers which have periodic decimal expressions (recall that a periodic decimal is one that takes the form $a_0.a_1 \ldots a_k b_1 \ldots b_l b_1 \ldots b_l \ldots$).
Let $S_3 = \mathbb{R} \times \mathbb{N}$. Which of these 3 sets are countable?
(a) all of them, (b) none of them, (c) S_1 and S_2, (d) S_2 only.

(ii) Which of the the following numbers is the highest common factor of 1001 and 49:
(a) 1 (b) 49 (c) 7 (d) 9 ?

(iii) Let $r_1 = 2 + \sqrt{3}$, $r_2 = (64)^{1/3}$, $r_3 = 0.1010010010000\ldots$. Which of these real numbers are irrational?
(a) all of them , (b) none of them, (c) r_1 and r_3 only , (d) r_1 only.

(iv) Let P_n be the set of all degree n polynomials with real coefficients. Let $D : P_{17} \to P_{16}$ be the map that sends a polynomial $p(x)$ to its derivative $p'(x)$. Which of the following is true?
(a) D is 1-1 but not onto,
(b) D is onto but not 1-1,
(c) D is neither 1-1 nor onto,
(d) D is a bijection

(v) Consider the following three statements:
P_1: If x is an upper bound for A and $x \in A$, then x is a least upper bound for A.
P_2: If x is a least upper bound for A then $x \in A$.
P_3: If $A \subseteq B$, x is a greatest lower bound for A and y is a greatest lower bound for B, then $y \leq x$.
Which of these 3 statements are true?
(a) all are true, (b) none are true, (c) P_1 and P_3 only, (d) P_2 and P_3 only.

(vi) Which one of the following cubics has roots $1+i$, $1-i$, and 2?
(a) $x^3 - 4x^2 + 6x + 4 = 0$
(b) $x^3 + 4x^2 - 6x + 4 = 0$
(c) $4x^3 - 6x^2 + 4x - 1 = 0$
(d) $ix^3 - 4ix^2 + 6ix - 4i = 0$.

(vii) Let $x = 64^{32}$, $y = 32^{64}$ and $z = 8^{100}$. Which of the following is true?
(a) $x < y < z$
(b) $x = y$ and $y < z$
(c) $x < z < y$
(d) $y < z < x$

(viii) Let $S = \mathbb{C}$ and define an relation on S by $a \sim b$ if and only if $|a - b| < 1$. Which of the following is true?

(a) $\sim$ is symmetric and reflexive but not transitive
(b) $\sim$ is symmetric but not reflexive or transitive
(c) $\sim$ is an equivalence relation
(d) $\sim$ is reflexive and transitive but not symmetric

(ix) How many complex numbers $z = x + iy$ with $x > 0$ satisfy $z^6 = 321$?

(a) 6 (b) 3 (c) 2 (d) infinitely many.

(x) Let r be the unique integer with $0 \leq r \leq 10$ such that $7^{37} \equiv r \mod 11$. Then r is equal to which of the following?

(a) 0 (b) 6 (c) 5 (d) 9.

SECTION B

2. (a) Prove using induction that every positive integer greater than 1 is equal to a product of prime numbers.

   (b) Give a careful statement of the Fundamental Theorem of Arithmetic.

   (c) Find all integer solutions x, y to the equation $x^2 = y^3$.

   (d) Prove that $\sqrt{n}$ is rational if and only if n is a perfect square.

3. (a) Let S be a non-empty subset of $\mathbb{R}$. Give the definition of a least upper bound for the set S.

   (b) Prove that S cannot have 2 different least upper bounds.

   (c) Prove that for any real number r, there exists a set of rationals having least upper bound equal to r.

   (d) Prove that for any positive integer n,

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

   (e) Let a and b be positive integers. Define the highest common factor, $\mathrm{hcf}(a, b)$ and the lowest common multiple $\mathrm{lcm}(a, b)$. Prove without using prime factorization that

$$\mathrm{lcm}(a, b) = \frac{ab}{\mathrm{hcf}(a, b)}.$$