BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2006

M1F (Test)

Foundations of Analysis

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (I) Say whether the following statements are true or false:

   (i) If $A = \{x \in \mathbb{R} \mid x^5 \geq 0\}$ and $B = \{x \in \mathbb{R} \mid 16 - x^2 > 0\}$ then

   $$A \cap B = \{x \in \mathbb{R} \mid 0 \leq x < 4\}.$$

   (ii) Let A, B, C and D be statements about real numbers such that $A \Rightarrow B$, $D \Rightarrow C$ and $C \Rightarrow B$. If A is true then D must be true.

   (iii) If $\omega = e^{\pi i/3}$ then $\omega + \omega^4 = 0$.

   (iv) If $a, b, c \in \mathbb{N}$ and c divides a^2b^2, then c^2 divides a^2b^2.

   (v) The function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = 1 - n^2$ is 1-to-1 (injective).

   (vi) The following is an equivalence relation on $\mathbb{C}$

   $$z \sim \omega \iff |z| - |\omega| \in \mathbb{Z}.$$

   (II) (i) Let E be the number of edges of a polyhedron with 6 vertices and 8 faces. Let V be the number of vertices of a cube. Which of the following is true?

   (a) $E > V$ (b) $E < V$ (c) $E = V$ (d) can't tell

   (ii) If $n \in \mathbb{N}$ is even then $\sqrt{2} - n \cdot 41\overline{42}$ is rational

   (a) always (b) sometimes (c) never

   (iii) Let $x = 2^{3/2}5^{1/2}4^{-1/4}\sqrt{10}$. Which of the following is true:

   (a) $x < 10$ (b) $x^3 \in \mathbb{Q}$ (c) $\frac{x}{\sqrt{5}} \in \mathbb{N}$ (d) $x^2 > 10x + 2$

   (iv) The binomial theorem states that $(1 - x)^n$ is equal to

   $$(a) \sum_{k=0}^{n} \binom{n}{k} (-x)^k \quad (b) \sum_{k=0}^{n} \binom{n}{k} x^k \quad (c) \sum_{k=0}^{n} \binom{n-k}{k} x^k \quad (d) \sum_{k=1}^{n} \binom{n}{k} x^k$$

   (v) How many complex numbers $z = x + iy$ with $x < 0$ satisfy $z^6 = 16$

   (a) 6 (b) 2 (c) 3 (d) 1 (e) infinitely many

   (vi) Which of the following statements is true for the set

   $$S = \{x \mid x \in \mathbb{R},\ 2x > x^2\}$$

   (a) LUB(S) exists but GLB(S) does not.
   (b) GLB(S) exists but LUB(S) does not.
   (c) Neither LUB(S) nor GLB(S) exists.
   (d) Both LUB(S) and GLB(S) exist.

   (vii) Let A and B be finite sets with $|A| = 5$ and $|B| = 10$. How many surjective (onto) maps $B \to A$ are there

   (a) 5^5 (b) 5^{10} (c) 10^{10} (d) some other positive number (e) none

SECTION B

2. (a) State the principle of induction.

   (b) Prove that for $n \geq 2$,

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

   (c) Let u_1, u_2, $u_3 \ldots$ be the sequence defined by

$$u_1 = 3,\ u_2 = 5 \quad \text{and} \quad u_n = 3u_{n-1} - 2u_{n-2} \text{ for } n \geq 3.$$

   Guess a simple formula for $u_n - 1$.
   Use (strong) induction to prove that your formula is correct.

3. (a) Fix $m \in \mathbb{N}$. Define what $a \equiv b \bmod m$ means for $a, b \in \mathbb{Z}$.

   (b) Prove $a \sim b \iff a \equiv b \bmod m$ is an equivalence relation on $\mathbb{Z}$.

   (c) Prove there do not exist positive integers n and r such that $n^2 \equiv r^2 + 2$ mod 4. (You may assume basic properties of modular arithmetic without proof.)

   (d) Find integers λ, μ such that $16\lambda + 334\mu = 10$.

   (e) Use the fact that $1001 = 13 \times 77$ to work out the remainder when 4003005008 is divided by 13.