UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2008

This paper is also taken for the relevant examination for the Associateship.

M1F

FOUNDATIONS OF ANALYSIS

Date: Friday, 16th May 2008 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Select which of the four possible answers to each part is correct. You are not required to give any further explanation of your reasoning and any such reasoning will not be read.

(a) Which of the following three functions are one-to-one?
$f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $f(m, n, p) = 2^m 3^n 5^p$ (do not include 0 in $\mathbb{N}$).
$g : \mathbb{R}^+ \to \mathbb{R}^+$ where $g(x) = x^{1/100}$ and $\mathbb{R}^+$ denotes the set of all positive real numbers.
$h : \mathbb{R} \to \mathbb{R}$ where $h(x) = x^{64} + 6x^{18} + 2$.
(i) f, g and h, (ii) f only, (iii) g only, (iv) f and g only.

(b) Which of the following three sets are countable?
$S_1 =$ all real numbers whose decimal expression begins 3.1415.
$S_2 =$ all irrational numbers between 0 and 1.
$S_3 =$ all real numbers that are integer powers of ten.
(i) S_1 and S_3 only, (ii) S_1, S_2 and S_3, (iii) S_3 only, (iv) S_2 only.

(c) Which of the following three statements are true?
P_1: a cubic with real coefficients always has three real roots counting multiplicity.
P_2: any odd degree polynomial with real coefficients has at least one real root.
P_3: there exists a real quintic polynomial with exactly one real root.
(i) P_1, P_2 and P_3, (ii) P_3 only, (iii) P_2 and P_3 only, (iv) P_1 and P_2 only.

(d) Which of the following three statements are true?
P_1: any subset $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound.
P_2: the least upper bound of a set $S \subset \mathbb{R}$ always belongs to S.
P_3: $LUB(S) = -GLB(-S)$.
(i) P_1, P_2 and P_3, (ii) P_3 only, (iii) P_1 and P_3 only, (iv) P_1 and P_2 only

(e) Let $S = \mathbb{C}$ and define a relation on S by $a \sim b$ if and only if $|a - b| < 2$. Which of the following is true?
(i) $\sim$ is an equivalence relation,
(ii) $\sim$ is reflexive and transitive, but not symmetric,
(iii) $\sim$ is reflexive but not transitive or symmetric,
(iv) $\sim$ is symmetric and reflexive but not transitive.

(f) Which of the following real numbers are irrational?
$x = \sqrt{27}$, $y = \sqrt{27}\sqrt{3}$, $z = \sqrt{27} + 1.\overline{2345}$.
(i) x and z only, (ii) x, y and z, (iii) x only, (iv) z only.

(g) Let n be a positive integer and ω be any nth root of unity. Which of the following statements is true?

P_1: if ω is an nth root of unity and n is even, then $-\omega$ is also an nth root of unity.

P_2: for any $\theta \in [0, 2\pi]$ there is some integer n so that $e^{i\theta}$ is an nth root of unity.

P_3: if ω is an nth root then ω^3 is also an nth root of unity.

(i) P_1, P_2 and P_3, (ii) P_3 only, (iii) P_1 and P_3 only, (iv) P_1 and P_2 only.

(h) Which of the following three statements are true?

P_1: If a, b and c are positive integers then $\left(a^b\right)^c = a^{b^c}$.

P_2: If a, b, and c are complex numbers then $(ab)c = a(bc)$.

P_3: If $a_1, \ldots a_n$ are complex numbers then $\overline{a_1 \ldots a_n} = \overline{a}_1 \ldots \overline{a}_n$.

(i) P_2 only, (ii) P_3 only, (iii) P_1, P_2 and P_3, (iv) P_2 and P_3 only.

(i) Which of the following three statements is true?

P_1: For any real number r there is a set S of irrationals with $GLB(S) = r$.

P_2: For any real number r there is a set T of rationals with $GLB(T) = r$.

P_3: If x is the least upper bound for a set S, then x is not a lower bound for S.

(i) P_2 only, (ii) P_1 and P_2 only, (iii) P_1, P_2 and P_3, (iv) P_1 and P_3 only.

(j) Let $f : S \to S$ and $g : S \to S$ be two functions defined on the set S. Which of the following statements are true?

P_1: If $f \circ g$ is onto then f is onto.

P_2: If $f \circ g$ is one-to-one then g is one-to-one.

P_3: if f and g are bijections then $f \circ g$ is a bijection.

(i) P_1, P_2 and P_3, (ii) P_3 only, (iii) P_1 and P_3 only, (iv) P_2 only.

2. (a) Prove that any positive integer greater than 1 can be written as a product of primes.

   (b) Use the previous part to prove that there are infinitely many prime numbers.

   (c) For any $n \in \mathbb{N}$ find a sequence of n consecutive positive integers none of which is prime.

   (d) Without multiplying out all the factors determine whether or not $13^2 \cdot 71 \cdot 103 = 17 \cdot 19 \cdot 53 \cdot 73$? Prove your answer stating any results you use and justifying carefully all claims you make.

3. (a) Find $\mathrm{hcf}(527, 901)$ and write it in the form $527s + 901t$ where s and t are integers.

   (b) Find the pair of integers s and t satisfying $527s + 901t = \mathrm{hcf}(527, 901)$ for which s is positive and as small as possible. Prove that there is no solution $s, t \in \mathbb{Z}$ for which $25 < s < 50$.

   (c) Let x and y be integers. Show that if $3|x^2 + y^2$, then $3|x$ and $3|y$.

   (d) Let x, y and $z \in \mathbb{Z}$. Show that $x^2 + y^2 + z^2$ cannot be of the form $8k + 7$ when:
   (i) exactly one of x, y, z is odd; and (ii) all of x, y, z are odd.
   Deduce that no integer of the form $8k+7$ is expressible as a sum of three integer squares.

4. (a) Define what it means for a real number x to be a least upper bound for a subset $S \subseteq \mathbb{R}$. When exactly does a subset of $\mathbb{R}$ have a least upper bound?

   (b) Let S be a nonempty subset of $\mathbb{R}$ which is bounded above and let $a > 0$. Define another subset of $\mathbb{R}$ by

   $$aS := \{as | s \in S\}.$$

   Prove that aS has a least upper bound u and that $u = aLUB(S)$.

   (c) Define a sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$ by

   $$a_1 = 1, \quad \text{and} \quad a_{n+1} = \sqrt{2 + \sqrt{a_n}} \quad \text{for } n \geq 1.$$

   Prove that $1 \leq a_n < a_{n+1} < 2$ for all $n \geq 1$.

   (d) Let (a_n) be the sequence of real numbers defined in part (c). Prove that the subset $S = \{a_n \,|\, n \in \mathbb{N}\} \subset \mathbb{R}$ has a least upper bound L.
   Let $\epsilon > 0$ be given. Prove that there exists some integer N so that

   $$-\epsilon < a_n - L < \epsilon \quad \text{for all} \quad n \geq N.$$