1. (a) Define what it means for a real number to be rational and for a real number to be irrational.

(b) Prove that if a is irrational and b is rational then $a+b$ is irrational.

(c) Prove that if $x = p+\sqrt{q}$, where p and q are rational, then for any $m \in \mathbb{N}$, $x^m = a+b\sqrt{q}$ for some rational numbers a and b.

(d) A real number α is said to be algebraic if there exists a polynomial P with *integer* coefficients $a_0, a_1, \ldots, a_n$ with $a_0 \neq 0$, such that

$$P(\alpha) = a_n\alpha^n + \ldots + a_1\alpha + a_0 = 0.$$

Prove that if $\alpha > 0$ is an algebraic number then so is $\sqrt{\alpha}$.

(e) Find a quadratic equation whose roots are $5+2\sqrt{6}$ and $5-2\sqrt{6}$. Hence, using part (d) or otherwise, prove that $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ are both algebraic numbers.

2. Let $n \in \mathbb{N}$ and r be a non-negative integer less than or equal to n. Define the binomial coefficient $\binom{n}{r}$ to be the number of r-element subsets of the set $S = \{1, 2, \ldots, n\}$.

(a) State the Multiplication Principle for an n-stage process. Use this principle to prove that the binomial coefficients as defined above are equal to

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

(You may assume that the number of arrangements of a set of n elements is $n!$.)

(b) State the Binomial Theorem.

(c) By considering $(1+x)^n.(1+x)^m$ prove that the binomial coefficients satisfy

$$\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} = \binom{n+m}{l}.$$

(d) By considering $(1+x)^n$ for two different choices of x prove that

$$\sum_{\substack{l \text{ odd} \\ 0 \leq l \leq n}} \binom{n}{l} = \sum_{\substack{l \text{ even} \\ 0 \leq l \leq n}} \binom{n}{l} = 2^{n-1}.$$

3. Let z be a complex number.

(a) Define the modulus, $|z|$, of z and give a geometric interpretation of it in terms of the Argand plane.

(b) Let w, $z \in \mathbb{C}$. Prove that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Interpret this fact in terms of the geometry of parallelograms. Draw a careful figure to illustrate your statement.

(c) Find the real and imaginary parts of $(\sqrt{3}-i)^{10}$ and $(\sqrt{3}-i)^{-7}$. For which values of n is $(\sqrt{3}-i)^n$ real?

(d) Find all the solutions of the equation

$$z^5 = -\sqrt{3} + i.$$

4. (a) Define what it means for a real number x to be a greatest lower bound for a subset S of $\mathbb{R}$. When exactly does a subset of $\mathbb{R}$ have a greatest lower bound?

(b) Find the greatest lower bounds and least upper bounds (if they exist) of the following sets:

$$S = \{x \mid 0 \leq x \leq \sqrt{2} \text{ and } x \in \mathbb{Q}\}, \quad T = \{\tfrac{1}{n} + (-1)^n \mid n \in \mathbb{N}\}.$$

Explain briefly your reasoning, making sure to explain why these why these bounds do or do not exist.

(c) Let S be a non-empty subset of $\mathbb{R}$ which is bounded above and let $a \in \mathbb{R}$. Define another subset of $\mathbb{R}$ by

$$a + S := \{a + S \,|\, x \in S\}.$$

Prove that $a + S$ has a least upper bound and that

$$LUB(a+S) = a + LUB(S).$$

(d) For both of the following statements, either prove that it is true or give a counterexample. If you give a counterexample, you should make sure to explain carefully why it is a counterexample.

(i) If a subset S of $\mathbb{R}$ contains only irrational numbers, and S has a least upper bound, then $LUB(S)$ is irrational.

(ii) Every real number is the least upper bound for some set of rational numbers.

5. (a) State the principle of strong induction.

(b) Define what it means for an integer $p \geq 2$ to be prime.

(c) Prove that every positive integer $m \geq 2$ is a product of prime numbers.

(d) Define what it means for two integers a and b to be congruent modulo a positive integer m.

(e) Calculate the remainder when 7^{15} is divided by 17.

(f) Suppose that a, b and c are integers, such that a and c are relatively prime. Prove that if c divides ab, then c divides b. Hence deduce that if p is prime and p divides ab then either p divides a or p divides b.
(*You may not assume the uniqueness of prime factorization to prove part (f)*).