Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M1F

Foundations of Analysis

Date: Tuesday, 16th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Define what it means for a real number to be *rational*.

   (b) Stating clearly any facts you use, decide which of the following numbers are rational.

   (i) $2\sqrt{3}$

   (ii) $\sqrt{2} + \sqrt{3/2}$

   (iii) $2\sqrt{18} - 3\sqrt{8} + \sqrt{4}$

   (c) Define what it means for a relation on a set to be *transitive*.

   (d) Decide which of the following relations on $\mathbb{R}$ are equivalence relations. For those that are, describe the equivalence class of 0; for those that are not, explain why.

   (i) $a \sim b \iff a^2 - b^2 \in \mathbb{Z}$

   (ii) $a \sim b \iff |a - b| < 1$

   (iii) $a \sim b \iff a^2 - b^2 \geq 0$

   (iv) $a \sim b \iff \dfrac{|a| + 1}{|b| + 1} \in \mathbb{Q}$

2. (a) Define the *modulus* $|z|$ and *complex conjugate* $\overline{z}$ of a complex number z.

   (b) Sketch the sets

$$L = \{z \mid |z - i| = |z + i|\} \quad \text{and} \quad M = \{p + qi \mid p, q \in \mathbb{Z}\}.$$

   (c) Let $\omega = 1 + i$. Write ω^7 in the form $re^{i\theta}$.

   (d) Again let $\omega = 1 + i$. For which positive integers n is $\omega^n = \overline{\omega}^n$?

   (e) Still with $\omega = 1 + i$, for each of the following integers decide (with proof) if $\omega^n + \sqrt{n} \in L \cup M$.

   (i) $n = 9$

   (ii) $n = 7$

   (iii) $n = 24$

3. (a) State the principle of strong induction.

   (b) Use simple induction to prove that $1^3 + 2^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$ for all positive integers n.

   (c) Define what it means for an integer to be *prime*.

   (d) Prove that every integer $n \geq 2$ is a product of primes.

   (e) Prove that if $m \geq 6$ then there exist integers $p, q \geq 0$ such that $m = 3p + 4q$.

4. (a) Consider a k-digit number $n = a_{k-1}a_{k-2}\dots a_1 a_0$. Prove that

$$n \equiv \sum_{j=0}^{k-1} (-1)^j a_j \bmod 11.$$

(b) Using the fact that $999 = 27 \times 37$, calculate the remainder obtained when one divides $9008007006005004003002001$ by 37.

(c) Let

$$A = \{1/n \mid n \in \mathbb{Z}, \quad n \equiv 1 \mod 7\} \quad \text{and} \quad B = \{1/m \mid m \in \mathbb{Z}, \quad m \equiv -1 \mod 9\}.$$

Prove that $A \cap B$ is an infinite set.

(d) Calculate the least upper bound $\mathrm{LUB}(A \cap B)$, and the greatest lower bound $\mathrm{GLB}(A \cup B)$.

5. (a) Let S and T be sets. Define what it means for a map $f : S \to T$ to be *surjective*, i.e. onto.

(b) Let $g : T \to U$ and $h : U \to V$ be surjective maps. Decide, with proof or counterexample, whether the following statements are true.

(i) The composite $h \circ g : T \to V$ is surjective.

(ii) There exists a surjective map $V \to T$.

(c) Find integers λ, μ such that $22\lambda + 15\mu = 2$ with $\mu > 100$.

(d) Define what it means for a map between sets to be injective, and construct an injective map $\mathbb{N} \to X$ where $X = \{\mu \mid 15\mu \equiv 1 \mod 22\}$.