UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
2005

This paper is also taken for the relevant examination for the Associateship.

M1F Foundations of Analysis

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Setter's signature	Checker's signature	Editor's signature
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1. (a) Define what it means for a real number to be irrational.

(b) Stating clearly any facts you assume, decide (with proof) which of the following numbers are rational and which are irrational. (You may assume that $\sqrt{3}$ is irrational.)

(i) $\sqrt{2}+\sqrt{3/2}$

(ii) $2+\sqrt{2}+\sqrt{3/2}$

(iii) $2\sqrt{18}-3\sqrt{8}+\sqrt{4}$

(c) Consider the following relation on the set $\mathbb{R}$:

$$a \sim b \iff 2ab \in \mathbb{Q}.$$

Stating your reasons clearly, decide if this relation is

(i) symmetric,

(ii) transitive,

(ii) reflexive.

(d) Let x be the real number whose decimal expansion is

$$x = 0 \cdot a_1 a_2 a_3 \ldots$$

where $a_n = 8$ if n is divisible by 3 and $a_n = 0$ otherwise. **Either** prove that x is irrational, **or** express it as a fraction.

(e) Let $I = \{x \in \mathbb{Q} \mid 0 < x < 1\}$ and $J = \{x \in \mathbb{Q} \mid 0 < x < 2\}$. Describe an injective (1-to-1) function $f : J \to I$.

2. (a) Define the *modulus* (absolute value) $|z|$ of a complex number z.

(b) Write $(1+i)(\sqrt{3}+i)$ in the form $x+iy$ and in the form $re^{i\theta}$.

(c) Deduce the exact value of $\sin(5\pi/12)$.

(d) Draw a clear sketch showing the sets of complex numbers
$C_1 = \{z : |z| = 1\}$ and $C_2 = \{z : |z-\sqrt{2}| = 1\}$.

(e) Show that if $z \in C_1 \cap C_2$ then $z^8 = 1$.

3. **(a)** Define what it means for an integer $p \geq 2$ to be *prime*.

(b) Prove that there are infinitely many prime numbers.
[You may assume the existence of prime factorisation.]

(c) Find all pairs of positive integers $x, y \in \mathbb{N}$ such that $x^3 = 8y^5$ and y is odd.
[You may quote the fundamental theorem of arithmetic without proof.]

(d) Let $A = \{\frac{1}{p} \mid p \text{ prime }\}$, let $B = \{x \mid x \in \mathbb{R},\, x^2 < 4\}$, let $C = A \cup B$ and $D = A \cap B$. Find the least upper bound (LUB) and greatest lower bound (GLB) for both C and D.

4. **(a)** State the principle of induction.

(b) Let $n \in \mathbb{N}$. Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

(c) The Fibonacci sequence (F_n) is defined as follows:

$$F_1 = F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

Prove that

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

for all $n \geq 1$.
[Hint: In the inductive step, you'll need to note that $(3 \pm \sqrt{5}) = \frac{1}{2}(1 \pm \sqrt{5})^2$.]

5. **(a)** Let $m \in \mathbb{N}$ and let $a, b \in \mathbb{Z}$. Define what it means to say that a is congruent to b modulo m, written $a \equiv b \mod m$.

(b) Prove that if hcf(a, m) divides b then there exists an integer x such that $ax \equiv b \mod m$.

(c) Find an integer $x > 100$ such that $75x \equiv 6 \mod 12$.

(d) Calculate the remainder when 7^{81} is divided by 15.

(e) Use the fact that $999 = 27 \times 37$ to calculate the remainder when 6005004003002001 is divided by 37.