UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M1F Foundations of Analysis

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Setter's signature	Checker's signature	Editor's signature
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1. **(a)** Define what it means for a real number to be rational.

(b) Find irrational numbers a and b such that $a + b$ is rational but ab is not.

(c) Prove the following statements.

(i) If $a \in \mathbb{N}$ is odd, then $\sqrt{2a}$ is irrational.

(ii) $\sqrt{60} + \sqrt{5/2}$ is irrational.

(d) Let x be the real number whose decimal expansion is

$$x = 2 \cdot \overline{114} = 2 \cdot 114114114\ldots$$

Write x as a fraction.

(e) Let y be the real number whose decimal expansion is

$$y = 1 \cdot a_1 a_2 a_3 \ldots$$

where $a_n = 2$ if $n = 2^m$ for some $m \in \mathbb{N}$ and $a_n = 1$ otherwise. Either prove that y is irrational, or else express it as a fraction.

2. **(a)** Define the conjugate $\overline{z}$ of a complex number $z \in \mathbb{C}$.

(b) Prove that $z + \overline{z}$ and $z\overline{z}$ are real for all $z \in \mathbb{C}$.

(c) Given $\omega \in \mathbb{C}$, describe a quadratic equation with real coefficients and roots ω and $\overline{\omega}$.

(d) Describe the roots of the equation $x^7 - 1 = 0$ in both polar form and the form $z = a + ib$. Draw the roots in the complex plane.

(e) Express the polynomial $x^7 - 1$ as the product of $(x - 1)$ and three real quadratics.

(f) Deduce that

$$(2 + 2\cos\frac{2\pi}{7})(2 + 2\cos\frac{4\pi}{7})(2 + 2\cos\frac{6\pi}{7}) = 1.$$

3. **(a)** State the principle of strong induction.

(b) Define what it means for an integer $p \geq 2$ to be *prime*.

(c) Prove that every positive integer $m \geq 2$ is a product of prime numbers.

(d) For each positive integer n, let $S_n = n^2 + (n+1)^2 + \cdots + (2n)^2$. Prove $S_n \geq 2n^3$.

(e) For each positive integer n, prove that $3^{2n} + 2^n 6$ is divisible by 7.

4. **(a)** Define what it means for two integers a and b to be congruent modulo m.

(b) Calculate the remainder when 7^8 is divided by 17.

(c) Prove that $7^{40} + 7^{16}$ is divisible by 17.

(d) Noting that $999 = 27 \times 37$, work out the remainder when 8009005003005008 is divided by 37.

(e) Find an integer t such that $78t \equiv 91 \bmod 143$.

(f) Find an integer x such that $2x^3 + 7x^2 + 4 = 0$, or else prove that no such integer exists. [*Hint: Work modulo 5.*]

5. (a) Define what it means for a binary relation $\sim$ on a set S to be an equivalence relation.

(b) For each of the following relations $\sim$ and sets S, decide if $\sim$ is reflexive, symmetric, and/or transitive. You do not need to give a proof when a property holds, but you should give a counterexample when it does not.

(i) $S = \mathbb{Z}$ and $x \sim y$ if $|x - y| \leq 2$.

(ii) $S = \mathbb{N}$ and $x \sim y$ if $xy = 2$.

(iii) $S = \mathbb{R}$ and $x \sim y$ if $x - y \in \mathbb{Q}$.

For each $\sim$ that is an equivalence relation, describe the equivalence class of 1.

(c) Find integers λ and μ so that $7\lambda + 5\mu = 1$ and $\lambda > 10$.

(d) Let $\sim$ be an equivalence relation on $\mathbb{Z}$ such that $x \sim x + 5$ and $x \sim x + 7$ for all $x \in \mathbb{Z}$. Prove that the equivalence class of 0 is $\mathbb{Z}$.