UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2008

This paper is also taken for the relevant examination for the Associateship.

M1A1

MECHANICS

Date: Monday, 19th May 2008 Time: 10am -12pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) A particle of mass m moves along the x-axis subject to a conservative force $F = F(x)$. Define the potential, $V(x)$ and derive the equation for the conservation of energy

$$\tfrac{1}{2}\, m\dot{x}^2 + V(x) = E \ .$$

(b) Show that in plane-polar co-ordinates, the acceleration has radial and transverse components

$$a_{\text{rad}} = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_{\text{trans}} = r\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} \ .$$

(c) Using the result of part (b), show that for motion in an isotropic central force, $\boldsymbol{F}(\boldsymbol{r}) = F(r)\hat{\boldsymbol{r}}$, the equation for the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{-1}{mh^2u^2}\ F\left(\frac{1}{u}\right) \ ,$$

where $u = \frac{1}{r}$ and h is the constant angular momentum per unit mass.

(d) A rocket of mass $m(t)$ moves along a straight line under the influence of an external force F . Exhaust fuel is expelled at constant velocity u relative to the rocket. Show that the equation of motion for the rocket is

$$m\dot{v} + u\dot{m} = F \ ,$$

where $\dot{v} \equiv \dfrac{dv}{dt}$ etc.

2. (a) Consider the equation of motion of a damped, forced oscillator

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2\, x = F_0 \cos\omega\, t \ ,$$

where ω , ω_0 , μ , F_0 are constants. Show that the steady-state, periodic solution is given by

$$x(t) = \frac{F_0}{[(\omega^2 - \omega_0^2)^2 + (2\mu\omega)^2]^{\frac{1}{2}}}\ \cos(\omega t + \Phi) \ .$$

and derive an expression for the phase Φ .

(b) Derive an expression for the work done by the applied force over one period τ and show this is equal to the work done against the resistive force.

3. (a) Consider the equation of motion of a projectile subject to air resistance

$$m\ddot{\boldsymbol{r}} = -mk\boldsymbol{v} - mg\boldsymbol{j} \ .$$

If the body is projected from the origin with velocity $\boldsymbol{u}$, show that at later times

$$\boldsymbol{r}(t) = \frac{-g}{k^2}(kt - 1 + e^{-kt})\boldsymbol{j} + \frac{\boldsymbol{u}}{k}(1 - e^{-kt}) \ .$$

(b) The body returns to Earth at at time T making an angle β to the horizontal determined as

$$\tan\beta = \frac{-\boldsymbol{v}(T).\boldsymbol{j}}{\boldsymbol{v}(T).\boldsymbol{i}} \ .$$

Show that

$$\frac{\tan\beta}{\tan\alpha} = \frac{e^{kT} - 1 - kT}{e^{-kT} - 1 + kT} \ ,$$

where α is the angle at take-off $(t = 0)$.

4. A particle rests on the top of a smooth fixed sphere of radius a and is slightly displaced. By writing the acceleration in polar co-ordinates, derive the expressions

$$\dot{\theta}^2 = \frac{2g}{a}(1 - \sin\theta) \ ,$$

and

$$R = mg\sin\theta - m\,a\,\dot{\theta}^2 \ ,$$

where R is the force of reaction on the particle and θ is the angle measured with reference to the horizontal. Hence show it leaves the surface of the sphere at height $\dfrac{2a}{3}$ above the centre with speed $\left(\dfrac{2ga}{3}\right)^{\frac{1}{2}}$.