1. A body of mass m moves along the x-axis subject to a potential $V(x)$. Show that the period τ of small oscillations about a stable equilibrium point x_0, is given by

$$\tau = 2\pi\sqrt{\frac{m}{V''(x_0)}} \quad .$$

Consider the potential

$$V(x) = \frac{x^2}{2} - \frac{x^4}{4} \quad .$$

If a body of unit mass is released from the point of stable equilibrium with velocity u, for what values of u does it

(i) oscillate

(ii) escape to infinity?

For the case $u = \frac{1}{\sqrt{2}}$ show that the position at time t, is given by

$$x = \frac{1 - e^{-\sqrt{2}t}}{1 + e^{-\sqrt{2}t}} \quad .$$

2. Consider the equation of motion for an undamped, forced oscillator

$$\ddot{x} + \omega_0^2 x = F_0 \cos \omega t \quad ,$$

where ω, ω_0, F_0 are constants.

If the body is released from the origin with zero velocity at $t = 0$, show that its position at later times is given by

$$x = \frac{2F_0}{\omega_0^2 - \omega^2} \quad \frac{\sin(\omega_0 + \omega)t}{2} \quad \frac{\sin(\omega_0 - \omega)t}{2} \quad .$$

Derive an expression for the time dependence of the mechanical energy $E(t)$ exactly at resonance, $\omega = \omega_0$.

3. Consider the motion of a plane pendulum of length a. Using appropriate intrinsic co-ordinates (s, ψ), derive the expression for the conservation of energy

$$E = \frac{1}{2} m\dot{s}^2 + 2mg\, a \sin^2 \frac{s}{2a} \quad ,$$

where g is the acceleration due to gravity. If the body is projected from $s = 0$ with velocity $\dot{s} = \sqrt{4ga}$, show that at later times

$$s = 2a\, \cos^{-1}\left(\text{sech}\sqrt{\frac{gt}{a}}\right) \quad .$$

What type of motion does this correspond to qualitatively?

4. A body of mass m moves in a central force field $F(r)$ directed away from the origin. Given that the radial and transverse accelerations in polar co-ordinates are

$$a_{rad} = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_{trans} = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad ,$$

show that

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2} \quad F\left(\frac{1}{u}\right) \quad ,$$

where $u = \frac{1}{r}$ and h is the angular momentum per unit mass.

A body is projected from the point $r = a$, $\theta = 0$ with transverse velocity $v_{trans} = \left(\frac{2}{a}\right)^{\frac{1}{2}}$ and zero radial velocity $(v_{rad} \equiv \dot{r} = 0)$, in a force field

$$F(r) = \frac{-m}{r^2}\left(1 + \frac{3a}{2r}\right) \quad .$$

Show that the equation for the orbit is

$$r = \frac{a}{2 - \cos\frac{\theta}{2}} \quad .$$

5. (a) A rocket of mass $m(t)$ moves in a straight line under the influence of an external force F. Exhaust gas is expelled at constant velocity u relative to the rocket. Show that the equation of motion for the rocket is

$$m\dot{v} + u\dot{m} = F \quad ,$$

where $\dot{v} \equiv \frac{dv}{dt}$, etc.

(b) Next consider the same rocket but moving around a circular track of radius a. Show that the equation of motion in polar co-ordinates is

$$\left(ma\ddot{\theta} + u\dot{m}\right)\widehat{\boldsymbol{\theta}} - ma\dot{\theta}^2\widehat{\mathbf{r}} = \mathbf{F} \quad ,$$

where $\widehat{\mathbf{r}}$ and $\widehat{\theta}$ are the radial and transverse unit vectors and $\mathbf{F}$ is the external force.