Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M1A1

Mechanics

Date: Wednesday, 17th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) A particle of mass m moves along the x-axis under the action of a conservative force $F = F(x)$. Derive the conservation of energy equation

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x)\,,$$

where $V(x)$ is the potential.

(b) Describe what is meant by a position of stable equilibrium x_0 and show that the period τ of small oscillations about it is given by

$$\tau = 2\pi\sqrt{\frac{m}{V''(x_0)}}\;.$$

(c) Sketch the potential $\qquad V(x) = \tfrac{1}{2}x^2 e^{-x^2}\,,$

identifying the positions of stable and unstable equilibrium.

If a particle of unit mass is projected with velocity u from the point of stable equilibrium, determine the values of u for which the body

(i) oscillates,

(ii) escapes to $x = +\infty$, $x = -\infty$.

2. (a) A particle of mass m moves along the x-axis subject to a force

$$F(x, \dot{x}, t) = -kx - 2m\mu\dot{x} + F_0 m\cos\omega t\,.$$

Show that the steady state solution to Newton's equation is

$$x = \frac{F_0}{[(\omega^2 - \omega_0^2)^2 + (2\mu\omega)^2]^{\frac{1}{2}}}\cos(\omega t + \Phi)\,,$$

where $\omega_0 = \sqrt{k/m}$ and Φ is the phase factor, to be determined.

(b) Determine the work, $W = 2m\mu\int\dot{x}dx$, done against the resistive force over one period (in the steady state), and show that this is equal to work done by the applied force (over the same time).

3. (a) A body of mass m is projected from the origin with velocity $\mathbf{u}$ (at angle α to the horizontal) subject to the force of gravity $-mg\mathbf{j}$ and air resistance $-mk\mathbf{v}$. Show that the position and velocity are given by

$$\mathbf{r}(t) \;=\; \frac{-g}{k^2}(kt-1+e^{-kt})\,\mathbf{j} \;+\; \frac{1-e^{-kt}}{k}\,\mathbf{u}\,,$$

$$\mathbf{v}(t) \;=\; \frac{-g}{k}(1-e^{-kt})\,\mathbf{j} \;+\; e^{-kt}\,\mathbf{u}\,.$$

(b) The body returns to Earth at angle $\beta\,(<\frac{\pi}{2})$ to the horizontal defined by

$$\tan\beta = \frac{-v_y(T)}{v_x(T)}\,,$$

where T is the time of flight, and v_x and v_y, are the x and y components to the velocity vector at time T. Show that β satisfies

$$\frac{\tan\beta}{\tan\alpha} = \frac{1-e^{kT}+kT}{1-e^{-kT}-kT}\,.$$

4. In plane polar co-ordinates (r,θ) the velocity and acceleration are given by

$$\mathbf{v} = \dot{r}\widehat{\mathbf{r}} + r\dot{\theta}\,\widehat{\boldsymbol{\theta}}\,,$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\widehat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\widehat{\boldsymbol{\theta}}\,.$$

(a) A bead of mass m is threaded on a smooth straight wire of length L that rotates in a horizontal plane with constant angular velocity ω about a vertical axis through one end (origin). From Newton's law show that the position r and force of reaction R satisfy

$$\ddot{r} - \omega^2 r = 0\,, \qquad (1)$$

and

$$R = 2m\,\omega\,\dot{r}\,, \qquad (2)$$

respectively.

(b) If the bead is projected with initial radial velocity $-u$ from the end of the wire $(r=L)$, determine the closest distance the particle gets to the origin.

(c) In a frame of reference that is rotating with constant angular velocity $\boldsymbol{\omega}$ Newton's second law reads

$$m\ddot{\mathbf{r}}_{rot} = \mathbf{F} - 2m\,\boldsymbol{\omega}\wedge\dot{\mathbf{r}}_{rot} + m\omega^2\mathbf{r}\,,$$

when we have assumed that $\boldsymbol{\omega}$ is perpendicular to $\mathbf{r}$.

Using a non-inertial frame, fixed to the rotating wire show that the same equations (1) and (2) derived in part (a) result from this non-inertial second law.

5. (a) A body of mass m moves in a central force field $\mathbf{F} = F(r)\widehat{\mathbf{r}}$. From Newton's equation of motion show that the orbit satisfies

$$\frac{d^2u}{d\theta^2} + u = \frac{-1}{h^2u^2m}\, F\,(\frac{1}{u})\,,$$

where $u = 1/r$ and h is the angular momentum per unit mass.

(b) A body moves in a central force field

$$F(r) = \frac{-m\mu}{r^2}(1 - \frac{\lambda}{r})\,,$$

with constants $\mu, \lambda > 0$.

Show that the equation of the orbit can be written

$$\frac{1}{r} = \frac{\mu}{h^2 + \lambda\mu} + A\cos\left(\sqrt{1 + \frac{\lambda\mu}{h^2}}\;\theta\right)\;.$$

(c) Assuming that the orbit is bounded, at what values of θ is the body closest and furthest from the origin? Hence sketch the path of bounded orbits.