1. (a) A particle P_1 moves on a circular path with constant speed, so that its position vector is $\underline{r}_1 = b\cos\Omega t\ \underline{i} + b\sin\Omega t\ \underline{j}$

(b, Ω positive constants).

A second particle P_2 moves on a circular path with the same centre and the same angular velocity, but with radius a ($> b$) and in the opposite direction.

Show that the path of particle P_2 relative to P_1 is an ellipse.

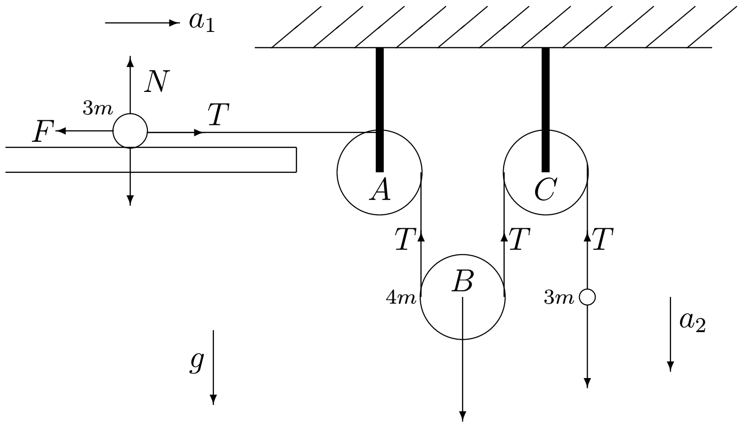
(b) Two particles Q_1 and Q_2 have position vectors $\underline{R}_1$ and $\underline{R}_2$ (respectively) and the same acceleration $\underline{g}$, where $\underline{g}$ is a constant vector.

At time $t = 0$: particle Q_1 is at the origin $\underline{O}$ and has velocity $\underline{U}_0$, while particle Q_2 is at position $\underline{R}_0$ and at rest.

Show that the minimum distance between Q_1 and Q_2 over all time is $|R_0 \sin\alpha|$, where α is the angle between $\underline{R}_0$ and $\underline{U}_0$, and that the minimum is achieved at a time $t > 0$ only when $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$.

2. One end of a light inextensible string is attached to a mass of $3m$, which rests on a rough horizontal table, the coefficient of friction between the mass and the table being μ. The string is at right angles to the edge of the table. It passes over a smooth fixed pulley A at the edge, under a smooth movable pulley B of mass $4m$ and over a smooth fixed pulley C, with a mass $3m$ attached to the other end of the string.

The portions of string are vertical between pulleys and the system is released from rest.



(a) If the accelerations of the $3m$, $3m$ masses are a_1, a_2 (as shown) state why the acceleration of the movable pulley is $\frac{1}{2}(a_1 - a_2)$ and write down the equations of motion for the three masses $3m$, $4m$, $3m$ in terms of the tension T in the string and the accelerations.

(b) When the coefficient of friction $\mu = \frac{1}{4}$, show that the tension in the string T is $\frac{39}{20}\ mg$, where g is the gravitational acceleration. By finding the accelerations determine whether the movable pulley i) rises or ii) falls.

(c) For what value of the coefficient of friction would the movable pulley be unaccelerated?

3. (a) A particle of mass m moves along the x axis under the action of a conservative force $F(x)$ for which the potential is $V(x)$. If x_0 is a position of stable equilibrium for which $V''(x_0) > 0$, show that the period T of small oscillations about x_0 is given by

$$T = 2\pi \left[\frac{m}{V''(x_0)} \right]^{1/2}$$

(b) If $V(x) = \frac{-Kx}{(x^2+4)}$ on $-\infty < x < +\infty$ and constant $K > 0$, show that there are two positions of equilibrium at $x = \pm 2$ and determine their stability. Find the period of small oscillations about the one that is stable.

(c) If the particle is projected from the stable position of equilibrium with velocity U in the x direction, write down the equation of conservation of energy and use it to find the range of values of U for which the particle

(i) escapes to $-\infty$

(ii) escapes to $+\infty$.

4. The displacement $x(t)$ of a particle from its equilibrium position satisfies the equation for a damped linear oscillator

$$\ddot{x} + p\dot{x} + \omega_0^2 x = 0$$

where $\omega_0 > \frac{p}{2} > 0$.

(a) Solve the equation for $x(t)$ in the form

$$x(t) = e^{-pt/2}(A\cos\Omega t + B\sin\Omega t)$$

with A, B, Ω constant, given that the motion is started from the position $x = a(> 0)$ at $t = 0$ from rest ($\dot{x}(0) = 0$).

(b) Show that the times t_n at which the particle passes through the equilibrium position $x = 0$ are given by $t_n = (n\pi + \alpha) \Big/ (w_0^2 - p^2/4)^{1/2}$ and $\cos\alpha = \frac{-p}{2w_0}$, $\sin\alpha = (w_0^2 - p^2/4)^{1/2}/w_0$.

(c) Find the velocity $\dot{x}(t_n)$ at these times.

(d) Sketch $x(t)$ as a function of time t.

5. (a) The position $\underline{r}$ of a point moving in a plane, is described by plane polar coordinates (r, θ).

Show that the velocity $\underline{v}$ and acceleration $\underline{a}$ are given by

$$\begin{aligned} \underline{v} &= \dot{r}\underline{e_1} + r\dot{\theta}\underline{e_2} \\ \underline{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e_1} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e_2} \end{aligned}$$

where $\underline{e_1}$ and $\underline{e_2}$ are the unit vectors

$$\begin{aligned} \underline{e_1} &= \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \\ \underline{e_2} &= -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}. \end{aligned}$$

(b) A particle of mass m moves in a plane under the action of a central force field $F(r) = \frac{-\mu m}{r^2}$ directed away from the origin $\underline{O}$, with μ a positive constant. By considering the acceleration components of m in plane polars show that the energy equation takes the form

$$\frac{1}{2}m\dot{r}^2 + \left[\frac{1}{2}m\frac{h^2}{r^2} - \frac{\mu m}{r}\right] = E$$

where E is the total energy and $h = r^2\dot{\theta} =$ constant.

(c) Given that the conic section $r = l/(1 + e\cos\theta)$ is a possible orbit for the mass m, find expressions for l and e in terms of E, h, m, μ.