1. (a) The position $\underline{r}$ of a point moving in a plane, is described by plane polar coordinates (r, θ).

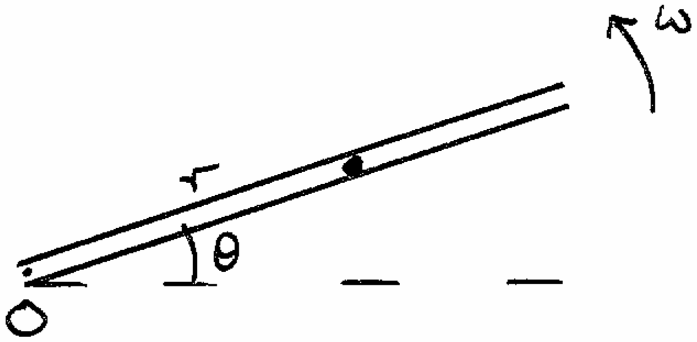
Show that the velocity $\underline{v}$ and acceleration $\underline{a}$ are given by

$$\begin{aligned} \underline{v} &= \dot{r}\underline{e}_1 + r\dot{\theta}\,\underline{e}_2 \\ \underline{a} &= (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_2 \end{aligned}$$

Where $\underline{e}_1$ and $\underline{e}_2$ are unit vectors

$$\begin{aligned} \underline{e}_1 &= \cos\theta\underline{i} + \sin\theta\underline{j} \\ \underline{e}_2 &= -\sin\theta\underline{i} + \cos\theta\underline{j} \end{aligned}$$

(b) A small bead of mass m slides inside a smooth straight tube of length $2l$ which rotates in a horizontal plane with constant angular velocity w about a vertical axis through the origin O at one end.



If the bead is released from left at $t = 0$, $r = l$ show that the bead leaves the tube after a time t_o, given by

$$t_o = \frac{1}{w}\cosh^{-1} 2.$$

Find the velocity of the bead at time t_o and its speed.

2. (a) A particle of mass m moves along the x-axis under a conservative force for which the potential is $V(x)$. If x_o is a position of stable equilibrium for which $V''(x_o) > 0$, show that the period T of small oscillations about x_o is given by

$$T = 2\pi \left[\frac{m}{V''(x_o)} \right]^{1/2}.$$

(b) If $V(x) = \frac{mkx}{(x^2+1)^2}$, where k is a positive constant, sketch the potential and determine the nature of the positions of equilibrium.

Show that the period T of small oscillations for the stable position is $2\pi \left(\frac{32}{27k\sqrt{3}} \right)^{1/2}$.

(c) If the particle is projected from its stable position with speed U in the x direction, show that it will oscillate if and only if $-\left(\frac{3k\sqrt{3}}{8} \right)^{1/2} < U < \left(\frac{3k\sqrt{3}}{8} \right)^{1/2}$.

Find U such that the particle will

(i). Escape to $-\infty$, (ii). Escape to $+\infty$.

3. A particle of mass m travelling along the x-axis satisfies the equation of motion

$$\ddot{x} + p\,\dot{x} + w_o{}^2\,x = \frac{F_o}{m}\,\cos(wt)$$

corresponding to a damped, driven oscillator.

[p, w_o, F_o, w are positive constants].

(a) Show that the steady-state (forced) solution is

$$x(t) = \frac{F_o}{m\left[(w^2 - w_o^2)^2 + p^2w^2\right]^{1/2}}\,\cos(wt + \Phi)$$

and determine the phase Φ.

(b) If p, w_o, F_o are kept constant, show that the forced solution (above) has largest amplitude when $w = \left(w_o{}^2 - \frac{p^2}{2}\right)^{1/2}$ in the case $w_o{}^2 > \frac{p^2}{2}$.

For what w is the amplitude largest when $w_o{}^2 < \frac{p^2}{2}$?

(c) In the case of no damping ($p = 0$) and exactly at *resonance* ($w = w_o$), show that the solution to the equation of motion is

$$x(t) = \frac{F_o\,t}{2w_o m}\,\sin(w_o\,t)$$

when the particle starts at the origin from rest.

4. A particle of mass m moves in a central force field $F(r)$ directed away from the origin O.

(a) Given that the radial and transverse acceleration components in polar coordinates are

$$a_{rad} = \ddot{r} - r\dot{\theta}^2 \text{ and } a_{trans} = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

respectively, show from the equations of motion that

$$\frac{d^2 u}{d\theta^2} + u = -\frac{1}{m\,h^2\,u^2}\,F\left(\frac{1}{u}\right).$$

Where $u = \frac{1}{r}$ and h is the angular momentum of the particle per unit mass.

(b) If the force is repulsive, such that $F(r) = \frac{km}{r^3}$, where k is a positive constant, and the particle is projected from the point $r = a$, $\theta = 0$ with radial and transverse velocity components U, V respectively, find the orbit and show that the particle approaches infinity along a direction given by

$$\tan(w\theta) = \left(w\frac{V}{U}\right)$$

with $w^2 = \left(1 + \frac{k}{a^2\,V^2}\right)$.

5. (a) A particle moves on a plane curve with intrinsic coordinates (s, ψ). Show that the acceleration at any point is given by

$$\underline{a} = \ddot{s}\,\widehat{\underline{T}} + \frac{\dot{s}^2}{\rho}\,\widehat{\underline{n}}$$

where the radius of curvature $\rho = \frac{ds}{dt}$ and $\widehat{\underline{T}}, \widehat{\underline{n}}$ are unit vectors tangent and normal to the curve at (s, ψ).

(b) A smooth wire in the shape of a parabola

$$y = a - k\,x^2 \qquad \text{with } a, k \text{ positive constants,}$$

stands in the vertical plane with the y-axis directed vertically upward.

A small bead of mass m slides on the wire and is released from rest from very near the highest point.

With what speed does the particle cross the line $y = 0$?

(c) Show that the reaction of the wire on the bead at this point is

$$R = \frac{m\,g}{(1 + 4\,k\,a)^{3/2}}.$$

$\left[\text{You may quote without proof } \rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \left|\left(\frac{d^2y}{dx^2}\right)\right|\right].$