UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY 2003
This paper is also taken for the relevant examination for the Associateship

M1A1 Mechanics

DATE: 21st May 2003 TIME: 10-12

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Question 5 is in multiple choice format. For each part **(i)** $-$**(x)** *a single alternative (a) – (e) should be specified. Ambiguous or illegible answers will be assumed to be wrong. No working is required to obtain full marks for that question.*

Calculators may not be used.

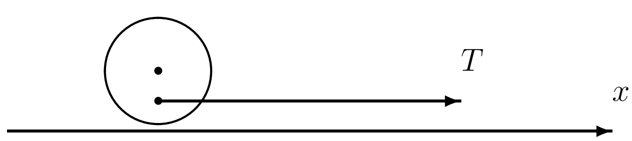
1. A rigid body, with momentum $\mathbf{P}$ and angular momentum about the origin $\mathbf{H}$, is acted upon by forces $\mathbf{F}_i$ at positions $\mathbf{r}_i$ for $i = 1, 2 \ldots N$. Define the total moment about the origin $\mathbf{G}$ and write down the governing equations for $\mathbf{P}$ and $\mathbf{H}$.

A circular wheel has radius a and mass m. At time t its centre is at position $(x(t),\, 0,\, a)$ with $x(0) = 0$. The wheel rolls in the x-direction along rough ground at $z = 0$, rotating with angular velocity $(0,\, \omega(t),\, 0)$ about its centre. Assuming that the wheel does not slip, relate x to ω, and find $x(t)$ if ω is constant.

At $t = 0$, a string is attached to the point $(0,\, 0,\, \frac{1}{2}a)$ and a constant force $(T,\, 0,\, 0)$ is applied by pulling on the string as in the diagram. Assuming that the wheel does not slip and has angular momentum $(0,\, I\omega(t),\, 0)$, where I is constant, show that the acceleration of its centre has magnitude

$$\frac{\frac{1}{2}a^2 T}{I + ma^2} \,.$$

If the wheel is stationary at $t = 0$, in which direction does it subsequently move?



2. A particle of mass m moves along the x-axis under the influence of a force $(F(x), 0, 0)$. Define the kinetic energy K and the potential U, and show that $K + U$ is constant.

Discuss whether or not the particle momentum $\mathbf{P}$ and angular momentum about the origin $\mathbf{H}$ are conserved.

When a rope of natural length L is extended by a positive amount x, the tension, T, is

$$T = \lambda \frac{x}{L} ,$$

where λ is a known constant. The tension is zero if x is negative. A man stands on a platform a height h above the ground and ties one end of such a rope to the platform. He attaches the other end to a harness on his body and then allows himself to fall towards the ground.

Show that his speed will be zero when he reaches the ground provided

$$\frac{L}{h} = 1 + \varepsilon - (2\varepsilon + \varepsilon^2)^{1/2} \qquad \text{where} \quad \varepsilon = \frac{mg}{\lambda} . \qquad (\#)$$

When a quadratic air resistance $-k\dot{x}^2$ is included in the problem, the corresponding equation for L (which you need not derive) is

$$\frac{1}{L}\left[h - L - \frac{1}{2k}\left(1 - e^{-2k(h-L)}\right)\right] = \varepsilon\left(1 - e^{-2kh}\right) .$$

Show that this expression is consistent with (#) in the limit $k \to 0$.

Turn over...

3. Two identical damped oscillators are linked together and forced so that their displacements $x(t)$ and $y(t)$ satisfy the equations

$$\begin{aligned}\ddot{x} + 2k\dot{x} + \omega^2 x + \alpha y &= F_0 \cos \omega t \\ \ddot{y} + 2k\dot{y} + \omega^2 y - \alpha x &= 0\end{aligned}$$

where k, ω, F_0 and α are constants. Indicate which are the damping, forcing and linkage terms in these equations.

Find a particular solution of the form

$$x = \Re e \left[x_0 e^{i\omega t} \right] \qquad\qquad y = \Re e \left[y_0 e^{i\omega t} \right] \ ,$$

where x_0 and y_0 are complex constants, and $\Re e$ denotes the real part.

Sketch the amplitude $|x_0|$ of the oscillation as a function of k when

(a) $\alpha = 0$

(b) $\alpha \neq 0$

and comment on any interesting features.

4. State Newton's law of gravitation.

A binary star consists of two particles of masses m_1 and m_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, which attract each other gravitationally.

Define $\mathbf{R}$, the position vector of their centre of mass, and show that it moves with constant velocity.

Show also that $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ satisfies the equation

$$\ddot{\mathbf{r}} = -\frac{G(m_1 + m_2)}{r^3}\,\mathbf{r}$$

where G is the gravitational constant and $r = |\mathbf{r}|$. Express this equation in polar form and derive the orbit equation

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{l} \qquad \text{where} \quad u = \frac{1}{r}\ ,$$

θ is the angle between $\mathbf{r}$ and an appropriate line, and l is a constant you should define.

Turn over...

5. *See the comments on the front cover relating to this question.*

(i) Which of the following **cannot** be a solution of the damped oscillator equation $\ddot{x} + a\dot{x} + bx = 0$, for suitable constants a and b?

(a) $2e^{-t} + e\,2^{-t}$, (b) te^{-t}, (c) $e^{-t}\cos 4t$, (d) $t\cos t$, (e) $2t + 3$.

(ii) A weight is attached to a fixed point $(0,\,0,\,a)$ by an inextensible string of length $\sqrt{2}a$. Gravity acts in the z-direction. The weight is moving steadily in a horizontal circle of radius a, but when it is at $(a, 0, 0)$ the string breaks. The subsequent motion of the weight is in

(a) the plane $y = 0$, (b) the plane $x = a$, (c) some other plane, (d) no single plane, (e) more information is required to determine the answer.

(iii) A particle with velocity $\mathbf{v}$ moves under a force such that

$$\dot{\mathbf{v}} = \mathbf{v} \wedge \mathbf{B} \qquad \text{where } \mathbf{B} \text{ is a constant vector.}$$

Which of the following statements is *false?*

(a) $\mathbf{v} \cdot \mathbf{B}$ is constant. (b) $|\mathbf{v}|$ is constant. (c) $\ddot{\mathbf{v}} = (\mathbf{v} \wedge \mathbf{B}) \wedge \mathbf{B}$. (d) If $\mathbf{v} \cdot \mathbf{B} = 0$ then the particle moves in a straight line. (e) $|\mathbf{v} \wedge \mathbf{B}|$ is constant.

(iv) A particle moves along the x-axis in a potential $U(x) = k(x^2 - a^2)(x^2 - b^2)$ where k, a and b are constant, with $a > b > 0$ and $k \neq 0$.

(a) If $k > 0$ there is a stable equilibrium at $x = a$. (b) If $k > 0$ there is an unstable equilibrium at $x = b$. (c) If $k < 0$ there is a stable equilibrium somewhere between $x = a$ and $x = b$. (d) If $k < 0$, a particle at $x = a$ will escape to infinity. (e) None of (a)–(d).

(v) Mike Tyson and Madonna have a tug-of-war. Each stands on rough ground pulling an end of a light inextensible rope. Madonna accelerates towards Tyson, but Tyson's centre of mass remains stationary. This is:

(a) because the rope exerts a greater force on Madonna than on Tyson. (b) because the ground exerts a greater horizontal force on Tyson than on Madonna. (c) because the ground exerts a smaller horizontal force on Tyson than on Madonna. (d) impossible by Newton's 3rd Law. (e) because the moment of the tension about Tyson's feet is greater than the moment about Madonna's feet.

(vi) A damped oscillator which is forced by the function $F_0 \cos \omega t$ for constant F_0 and ω :

(a) always ends up stationary if the damping exceeds critical. (b) eventually oscillates with an amplitude independent of ω. (c) theoretically reaches an infinite amplitude if ω is the resonant frequency. (d) moves with constant velocity if $\omega = 0$. (e) none of (a)–(d) is true.

(vii) At $t = 0$ a bungee jumper is stationary at height $x = 1$. Subsequently,

$$\ddot{x} = -x \qquad \text{for} \quad x < 0, \qquad \ddot{x} = -1 \qquad \text{for} \quad x > 0 \, .$$

In her subsequent motion, she reaches the lowest point:

(a) $x = -\pi$ at $t = \sqrt{2} + \pi$. (b) $x = -\sin^{-1}\sqrt{2}$ at $t = \sqrt{2} + \frac{1}{2}\pi$. (c) $x = 1 - \sqrt{2}$ at $t = \ln 2 + \pi$. (d) $x = -\sqrt{2}$ at $t = \sqrt{2} + \frac{1}{2}\pi$. (e) $x \to -\infty$ as $t \to \infty$.

(viii) Which of the following pairs **does not** have the same physical dimensions?

(a) potential energy and the rate of change of angular momentum. (b) the moment of a force and kinetic energy. (c) angular velocity and frequency of small oscillations. (d) the Coriolis force and weight of a body on the moon. (e) Each pair in (a)-(d) has the same dimensions.

(ix) A particle of mass m is moving with velocity $\mathbf{v}$ in a frame rotating with constant angular velocity $\boldsymbol{\omega}$. If W is the rate at which the Coriolis force $\mathbf{F}_{cor} = -2m\,\boldsymbol{\omega} \wedge \mathbf{v}$ does work in this frame, then:

(a) $W = 0$ always. (b) $W > 0$ in the Northern hemisphere, and $W < 0$ in the Southern hemisphere. (c) $W \geq 0$ always, and usually $W \neq 0$. (d) W can be strictly positive for motion in an East/West direction, but is always zero for motion in a North/South direction. (e) none of (a) – (d).

(x) The moon has mass 7×10^{22}kg and is a distance 4×10^{8}m from the earth. The Queen's Tower can be treated as a point mass of 10^{6}kg. Roughly how many metres must you be from the Queen's Tower for the gravitational force of the tower and the moon to be comparable?

(a) 10^{-8} (b) 10^{-3} (c) 1 (d) 10^{2} (e) 10^{4}.