Paper Number(s): **E2.5**
ISE2.7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2000

EEE/ISE PART II: M.Eng., B.Eng. and ACGI

SIGNALS AND LINEAR SYSTEMS

Wednesday, 14 June 2000, 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Time allowed: 2:00 hours

Corrected Copy

Examiners: Prof A.G. Constantinides, Dr J.A. Chambers

1. The first stage of a digital signal processing system has the Finite Impulse Response system transfer function

$$H(z) = 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} + \ldots + \alpha^{(N-1)} z^{-(N-1)}$$

where N is a positive integer.

a) Write down the difference equation for the system directly from the above form of $H(z)$. **(5 marks)**

b) By employing the sum of a geometric series, or otherwise, write $H(z)$ in the form of a ratio of two polynomials in z^{-1}. Hence derive the difference equation for the system from the new form of $H(z)$. Comment on the result. **(12 marks)**

c) Comment on the allowable values for α when N is large but finite, and when N is infinite. **(4 marks)**

d) Two new transfer functions $G_1(z)$ and $G_2(z)$ are formed from $H(z)$ by setting $\alpha = 1$ and $\alpha = -1$ respectively. Determine appropriate expressions for their amplitude and phase responses. **(12 marks)**

2. The frequency dependent gain of a system is measured and plotted in Figure 1. The vertical axis is the amplifier gain in dB while the horizontal axis is the logarithm, to the base 10, of the angular frequency.

a) Justify the expectation that the order of the numerator of the transfer function can be made the same as the order of the denominator. **(3 marks)**

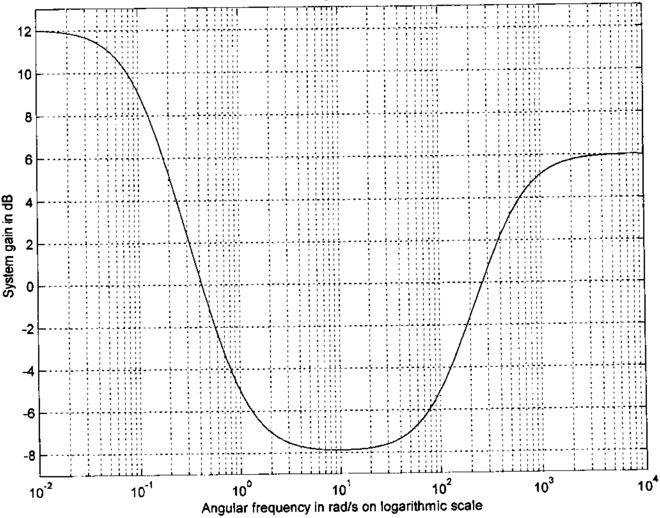
b) On the given graph by locating appropriate 3 dB points, or otherwise, select four break frequencies and draw on the same diagram the Bode linear approximation. Explain fully all steps you take. **(8 marks)**

c) Determine a minimum phase transfer function $H_1(s)$ and a maximum phase transfer function $H_2(s)$ that have the given response on the basis of your choice above. Make sure that all relevant constant factors are included. **(12 marks)**

d) Construct for $H_1(s)$ the Bode approximation to the phase response. **(10 marks)**

(NOTE: Additional copies of Figure 1 are available)

Figure 1


12
10
8
6
4
2
0
-2
-4
-6
-8
System gain in dB
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Angular frequency in rad/s on logarithmic scale

3. Explain what is meant by the terms *linear systems, transfer function,* and *impulse response.* **(3 marks)**

Two continuous-time systems are described by the following differential equations

System S1: $\dfrac{d^2 y_1(t)}{dt^2} + 2\alpha \dfrac{dy_1(t)}{dt} + \omega_0^2 y_1(t) = 2\alpha x(t)$

System S2: $\dfrac{d^2 y_2(t)}{dt^2} + 2\alpha \dfrac{dy_2(t)}{dt} + \omega_0^2 y_2(t) = 2\alpha \dfrac{d^2 x(t)}{dt^2}$

where α and ω_0 are real positive constants

$x(t)$ is the input signal

$y(t)$ is the output signal

a) By assuming zero initial conditions and using the Laplace transform determine the transfer functions $H_1(s)$ and $H_2(s)$ respectively. **(10 marks)**

b) Find the poles and zeros of $H_1(s)$ and $H_2(s)$. **(2 marks)**

c) Sketch the amplitude responses of $H_1(s)$ and $H_2(s)$ and provide arguments to support the opinion that one is lowpass while the other is highpass. **(10 marks)**

d) Determine the outputs $y_1(t)$ and $y_2(t)$ when the input is $x(t) = \cos(\omega_0 t)$. **(8 marks)**

4. The Laplace transform of a causal signal $x(t)$ is given as $X(s)$. Determine the Laplace Transform of $tx(t), \quad t \geq 0$. **(6 marks)**

A causal linear time-invariant system has a transfer function

$$H(s) = \frac{s+4}{s^2+5s+6}$$

a) Determine the differential equation that relates the input $x(t)$ and the output $y(t)$. **(5 marks)**

b) Determine the impulse response $h(t)$. **(10 marks)**

c) Without using the convolution relationship find the output $y(t)$ when the input is $x(t) = e^{-4t} - te^{-4t}$ for $t \geq 0$. **(12 marks)**

5. A causal discrete-time signal $\{x(n)\}$ is given from which a new discrete-time signal $\{y(n)\}$ is generated according to the relationship

$$y(n) + ay(n-1) = bx(n) + x(n-1)$$

The parameters a and b are real.

a) Derive the transfer function $H(z)$ of the above operation. **(5 marks)**

b) Determine the range of values for a and b to make $H(z)$ stable. **(5 marks)**

c) Show that when $a = b$ the amplitude response of $H(z)$ satisfies the condition $\left|H(e^{j\theta})\right| = 1$ for all θ. **(11 marks)**

d) By using the difference equation above, or otherwise, determine the impulse response $\{h(n)\}$ of $H(z)$ when $a = b$ for $n = 0,1,2,3,4$. Show that $h(n) = -ah(n-1)$ for $n \geq 2$ and hence derive a general expression for $\{h(n)\}$. **(12 marks)**

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EE2 & ISE2 2000

Q.1.

a) From the definition of the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\text{Output } z\text{-transform}}{\text{Input } z\text{-transform}}$$

we have

$$Y(z) = X(z) + \alpha\left[z^{-1}X(z)\right] + \alpha^2\left[z^{-2}X(z)\right] + \cdots + \alpha^{(N-1)}\left[z^{-(N-1)}X(z)\right]$$

and hence by inversion

$$y(n) = x(n) + \alpha x(n-1) + \alpha^2 x(n-2) + \cdots + \alpha^{N-1} x(n-N+1)$$ 5

b) Write $H(z) = 1 + (\alpha z^{-1}) + (\alpha z^{-1})^2 + \cdots + (\alpha z^{-1})^{N-1}$

$$= \frac{1-(\alpha z^{-1})^N}{1-\alpha z^{-1}}$$ using the sum of a geometric series

Hence

$$Y(z)\left[1-\alpha z^{-1}\right] = X(z)\left[1-(\alpha z^{-1})^N\right]$$

or $y(n) = \alpha y(n-1) + x(n) - \alpha^N x(n-N)$

- This is a recursive form in that it requires both the input <u>and</u> the past output values.
- The non-recursive form needs $(N-1)$ multipliers (as it stands) for its realisation while the recursive form requires only two multipliers. 12

c) When N is finite, α can take any value as the impulse response is of finite duration. However when N is infinite then the transfer function has a physical meaning if $|\alpha| < 1$ when $H(z)$ is also stable 4

d) $$G_1(z) = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^{-N/2}\left[z^{N/2} - z^{-N/2}\right]}{z^{-\frac{1}{2}}\left[z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right]}$$

and for $z = e^{j\theta}$

$$G_1(e^{j\theta}) = e^{-j\left(\frac{N-1}{2}\right)\theta} \cdot \frac{\sin N\theta/2}{\sin \theta/2}$$

i.e. Amplitude response $= \dfrac{\sin N\theta/2}{\sin\theta/2}$; Phase $= -\dfrac{(N-1)\theta}{2}$

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For $G_2(z)$ we have $\alpha = -1$ and hence

$$G_2(z) = \frac{1+z^{-N}}{1+z^{-1}} = \frac{z^{-N/2}\left[z^{N/2} + z^{-N/2}\right]}{z^{-1/2}\left[z^{1/2} + z^{-1/2}\right]}$$

and thus

$$\text{Amplitude response} = \frac{\cos N\theta/2}{\cos \theta/2}$$

and

$$\text{Phase response} = -\left(\frac{N-1}{2}\right)\theta$$

12

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Q.2.

a) For very high frequencies it is observed from the plot that the gain tends to a constant value and hence for rational transfer functions this means that the order of the numerator is the same as the order of the denominator 3

b) From the flat portions of the response the 3dB are located easily as indicated by ①, ②, ③ and ④ in the figure. These are the break frequency points and are given by

0.1, 1, 100, 500

c) they yield the transfer function (minimum phase) 8

$$H(s) = A . \frac{(s+1)(s+100)}{(s+0.1)(s+500)}$$

As $s \to \infty$ $H(s) \to A$ and from the figure
$6dB = 20\log_{10} |H(\infty)|$ i.e. $|H(\infty)| = 2$
Hence minimum ϕ

$$H_1(s) = 2 \frac{(s+1)(s+100)}{(s+0.1)(s+500)}$$

while maximum ϕ is 12

$$H_2(s) = 2 \frac{(s-1)(s-100)}{(s+0.1)(s+500)}$$

d) The phase response of $H_1(s)$ is

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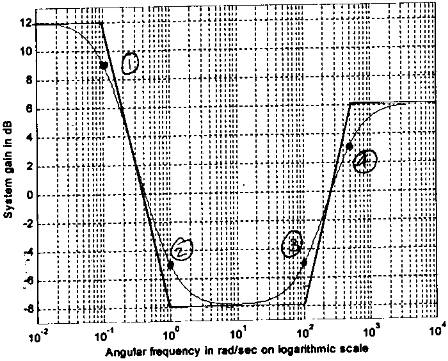
Figure 1



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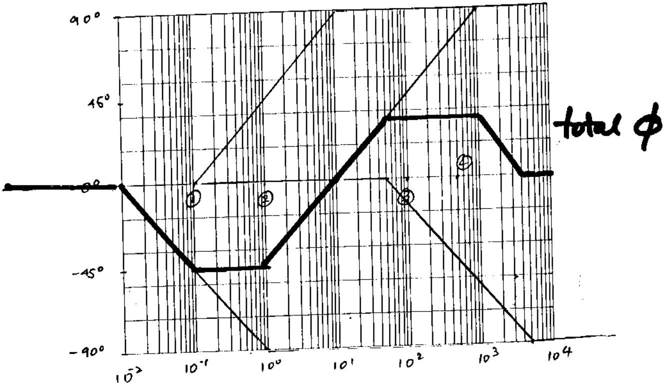

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Q.3.

Expected points are either w.r.t. continuous or discrete - time systems

- Linearity: $S\{\alpha x_1(t) + \beta x_2(t)\} = \alpha S\{x_1(t)\} + \beta S\{x_2(t)\}$
- Output Transform / Input Transform = T.F
- Output when input is either $\delta(t)$ (cont. t) or $\delta(n)$ (discrete t) is impulse response 3

a) From the Laplace transforms of the two D.E we have

S_1: $s^2 Y(s) + 2\alpha s Y(s) + \omega_0^2 Y(s) = 2\alpha X(s)$

S_2: $s^2 Y(s) + 2\alpha s Y(s) + \omega_0^2 Y(s) = 2\alpha s^2 X(s)$

Hence $H_1(s) = \dfrac{2\alpha}{s^2 + 2\alpha s + \omega_0^2}$

$$H_2(s) = \frac{2\alpha s^2}{s^2 + 2\alpha s + \omega_0^2}$$

for $\omega_0^2 = \alpha^2 + \beta^2$

$$H_1(s) = \frac{2\alpha}{s^2 + 2\alpha s + \alpha^2 + \beta^2} = \frac{2\alpha}{(s+\alpha)^2 + \beta^2}$$

and $H_2(s) = \dfrac{2\alpha . s^2}{(s+\alpha)^2 + \beta^2}$

Hence	Poles	zeros
$H_1(s)$	$-\alpha \pm j\beta$	2 at $s = \infty$
$H_2(s)$	$-\alpha \pm j\beta$	2 at $s = 0$

10

Amplitude Responses:

1) $|H_1(j\omega)| \to 0$ as $s \to \infty$ and is equal to $\frac{1}{\omega_0}$ at $s = j\omega_0$ 2
and $2\alpha/\omega_0^2$ at $s = 0$

2) $|H_2(j\omega)| \to 2\alpha$ as $s \to \infty$ and is equal to 0 at $s = 0$
and ω_0 at $s = j\omega_0$.

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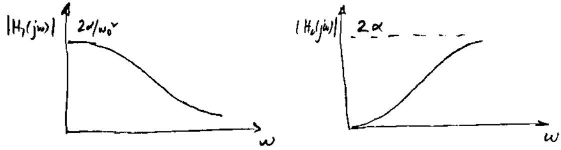
Thus plots are expected to be



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c) $y_1(t) = \frac{1}{2}\left[e^{j\omega_0 t} \cdot H_1(j\omega_0) + e^{-j\omega_0 t}\, H_1(-j\omega_0)\right]$

But $H_1(j\omega_0) = \frac{1}{j\omega_0}$ and $H_1(-j\omega_0) = -\frac{1}{j\omega_0}$

Hence

$$y_1(t) = \frac{1}{\omega_0} \cdot \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] = \frac{\sin \omega_0 t}{\omega_0}$$

$y_2(t) = \frac{1}{2}\left[e^{j\omega_0 t} \cdot H_2(j\omega_0) + e^{-j\omega_0 t} \cdot H_2(-j\omega_0)\right]$

But $H_2(j\omega_0) = \frac{-2\alpha\omega_0^2}{2\alpha j\omega_0} = j\omega_0$ and $H_2(-j\omega_0) = -j\omega_0$

Hence

$$y_2(t) = +j\omega_0 \cdot \frac{1}{2}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$= -\omega_0 \cdot \sin \omega_0 t$$

8

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Q.4.

Set $X(s) = \int_0^\infty x(t) e^{-st} \, dt$

Observe that $$\frac{dX(s)}{ds} = \int_0^\infty x(t)(-t) e^{-st} dt$$
$$= -\int_0^\infty t\, x(t) e^{-st} dt$$

and hence $\mathcal{L}\{t\, x(t)\} = -\frac{d}{ds} X(s)$ 6

a) Write
$$\frac{Y(s)}{X(s)} = \frac{s+4}{s^2+5s+6}.$$

Thus
$$s^2 Y(s) + 5sY(s) + 6Y(s) = sX(s) + 4X(s)$$
and hence by inversion we obtain
$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 4x(t) + \frac{dx(t)}{dt}$$ 5

b) The impulse response $h(t)$ is the output $y(t)$ when $x(t) = \delta(t)$

i.e. $$\frac{H(s)}{1} = \frac{s+4}{s^2+5s+6}$$

or $$H(s) = \frac{s+4}{(s^2+5s+6)} = \frac{s+4}{(s+2)(s+3)} = \frac{(-2+4)/1}{s+2} + \frac{(-3+4)/-1}{s+3}$$
$$= \frac{2}{s+2} - \frac{1}{s+3}$$

by inversion we have
$$h(t) = 2e^{-2t} - e^{-3t}$$ 10

c) Take the Laplace transform of $x(t)$

$$X(s) = \frac{1}{s+4} + \frac{d}{ds}\left(\frac{1}{s+4}\right) = \frac{1}{s+4} - \frac{1}{(s+4)^2}$$
$$= \frac{1}{s+4}\,\frac{s+4-1}{s+4} = \frac{s+3}{(s+4)^2}$$

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Now form

$$Y(s) = \frac{s+4}{s^2+5s+6} \cdot \frac{s+3}{(s+4)^2} = \frac{\cancel{(s+4)}\cancel{(s+3)}}{(s+2)\cancel{(s+3)}(s+4)^{\cancel{2}}}$$

$$= \frac{1}{(s+2)(s+4)} = \frac{1/2}{s+2} - \frac{1/2}{s+4}$$

and hence

$$y(t) = \tfrac{1}{2}\left[e^{-2t} - e^{-4t} \right]$$ 12

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Q5.

$y(n) + a\,y(n-1) = b\,x(n) + x(n-1)$

Take z-transforms of both sides

$Y(z) + a z^{-1} Y(z) = b X(z) + z^{-1} X(z)$

and hence the transfer function

$$H(z) = \frac{\text{Output z-transform}}{\text{Input z-transform}} = \frac{b + z^{-1}}{1 + a z^{-1}} = \frac{bz+1}{z+a}$$ 5

b) There is a zero at $z = -1/b$ – It plays no part in the stability of $H(z)$

There is a pole at $z = -a$ which determines the stability of $H(z)$ and hence $|a| < 1$ 5

c) If $a = b$ then

$$H(z) = \frac{a + z^{-1}}{1 + a z^{-1}} = z^{-1} \frac{[1 + a z]}{[1 + a z^{-1}]}$$

and since a is real then

$$[1 + a e^{-j\theta}]^* = [1 + a e^{+j\theta}] = A^* \text{ (say)}$$

ie.

$$H(z)\Big|_c = e^{-j\theta} \cdot \frac{A^*}{A} \quad . \text{ But } |e^{-j\theta}| = 1 \text{ and } \left|\frac{A^*}{A}\right| = 1$$

i.e $|H(z)| = 1$ when $z = e^{j\theta}$ 11

d) From the difference equation

$$y(n) = -a\,y(n-1) + a\,x(n) + x(n-1)$$

when $x(n) = \delta(n)$ then $y(n) = h(n)$

ie. $n = 0$ $h(0) = -a \cdot 0 + a \cdot 1 + 0 = a$

$n = 1$ $h(1) = -a\,h(0) + a \cdot 0 + 1 = 1 - a^2$

$n = 2$ $h(2) = -a h(1) + a \cdot 0 + 0 = -a(1-a^2)$

$n = 3$ $h(3) = -a\,h(2) + 0 + 0 = a^2(1-a^2)$

$n = 4$ $h(4) = -a \cdot a^2(1-a^2) = (-a)^3(1-a^2)$

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Hence

$$h(n) = -a \cdot h(n-1) \qquad n \geq 2$$

Thus if $h(n) = (-a)^{n-1} + (-a)^{n+1} \qquad n \geq 2$

then

$$\begin{aligned} h(n+1) &= (-a)^{n} + (-a)^{n+2} \\ &= (-a)\left[(-a)^{n-1} + (-a)^{n+1}\right] \\ &= (-a)\, h(n) \end{aligned}$$

i.e. the formula is true for $h(n+1)$ if it is assumed true for n.

It is also true for $n = 2(3)$ and hence it is true for all n.

11.

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