MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

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PAPER ISE 2.6

QUESTION

SOLUTION 1

(i) $$\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt = \left[\cancel{f(t)e^{-i\omega t}}\right]_{-\infty}^{+\infty} + i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$= i\omega\,\hat{f}(\omega) \qquad \boxed{a_1(\omega) = i\omega}$$ 2

(ii) $$\int_{-\infty}^{\infty} f''(t)e^{-i\omega t}dt = \left[\cancel{f'(t)e^{-i\omega t}}\right]_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt = i\omega\,\widehat{f'(t)}(\omega)$$

$$= -\omega^2\hat{f}(\omega) \qquad \boxed{a_2(\omega) = -\omega^2}$$ 2

(iii) $$\frac{d\hat{f}(\omega)}{d\omega} = -i\int_{-\infty}^{\infty} t e^{-i\omega t}f(t)dt \quad \text{so} \quad \widehat{tf(t)}(\omega) = i\,d\hat{f}(\omega)/d\omega$$ 2

$$\int_{-\infty}^{\infty} tf'(t)e^{-i\omega t}dt \qquad \frac{d}{d\omega}\left(\widehat{f'(t)}(\omega)\right) = -i\int_{-\infty}^{\infty} tf'(t)e^{-i\omega t}dt$$

$$\text{so} \int_{-\infty}^{\infty} tf'(t)e^{-i\omega t}dt = i\frac{d}{d\omega}\left(\widehat{f'(t)}(\omega)\right) = i\frac{d}{d\omega}\left(i\omega\hat{f}(\omega)\right)$$

$$= -\hat{f}(\omega) - \omega\frac{d\hat{f}}{d\omega}(\omega)$$ 4

$$= -\hat{f}(\omega) + i\,a_1(\omega)\frac{d\hat{f}}{d\omega}$$

$$y'' + 2ty' + 2y = 0$$

F.T. wrt

$$\Rightarrow \quad -\omega^2\hat{y}(\omega) + 2\left(-\hat{y} - \omega\,\hat{y}'(\omega)\right) + 2\hat{y} = 0$$

$$\Rightarrow \quad \omega\hat{y} + 2\hat{y}'(\omega) = 0$$ 3

$$\hat{y}' = -\frac{\omega}{2}\hat{y}$$ 2

$$\Rightarrow \quad \hat{y} = Ae^{-\omega^2/4} \qquad (A \text{ constant})$$

$$\Rightarrow \quad y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{y}(\omega)e^{i\omega t}d\omega =$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4} + i\omega t}d\omega = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4}(\omega - 2it)^2 - \frac{t^2}{4}}d\omega$$ 3

$$= \frac{A}{2\pi}e^{-t^2/4}\int_{-\infty}^{\infty} e^{-u^2}2\,du \qquad \omega - it = 2u$$

$$= \frac{A}{\sqrt{\pi}}e^{-t^2/4} = e^{-t^2/4} \quad \text{if } y(0) = 1$$

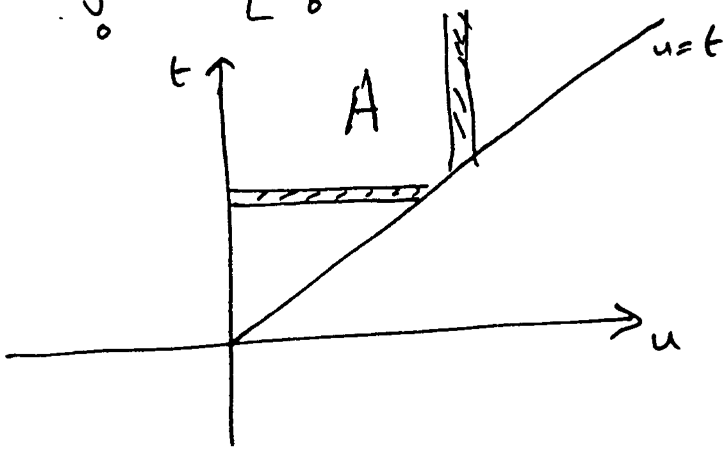
Direct Calculation: $y' = -\frac{t}{2}e^{-t^2/4}$, $y'' = \frac{t^2}{4}e^{-t^2/4} - \frac{1}{2}e^{-t^2/4}$ o.k. 2

Setter : ATKINSON Setter's signature : C. Atkinson

Checker : C J RIDLER-ROWE Checker's signature : C J Ridler Rowe

Treat

$$I = \int_0^\infty e^{-st}\left[\int_0^t f(t-u)\,g(u)\,du\right]dt$$ as a double integral.



$$= \iint_A e^{-st} f(t-u)\,g(u)\,du\,dt$$ 4

$$= \int_0^\infty g(u)\,du\left[\int_u^\infty e^{-st} f(t-u)\,dt\right]$$

$$= \int_0^\infty g(u)\,du\left[\int_0^\infty e^{-s(u+t_1)} f(t_1)\,dt_1\right] \quad t-u=t_1$$ 4

$$= \int_0^\infty g(u)\,e^{-su}\,du \int_0^\infty e^{-st_1} f(t_1)\,dt_1 = \bar{g}(s)\bar{f}(s)$$

L.T. of integral eqn using Convolution Thm 4

$$\Rightarrow \bar{y}(s) = \frac{1}{s+3} - \bar{y}(s)\,\frac{1}{(s+1)}$$

$$\Rightarrow \bar{y}(s)\frac{(s+2)}{(s+1)} = \frac{1}{(s+3)} \Rightarrow \bar{y}(s) = \frac{(s+1)}{(s+2)(s+3)}$$ 3

$$= \frac{-1}{(s+2)} + \frac{2}{(s+3)}$$ Partial Fractions

(Invert) $\Rightarrow y(t) = 2e^{-3t} - e^{-2t}$ 2

Direct Substn

R.H.S. $= e^{-3t} - 2\int_0^t e^{-3(t-u)-u}\,du + \int_0^t e^{-2(t-u)-u}\,du$

$$= e^{-3t} - 2e^{-3t}\left[\frac{e^{2u}}{2}\right]_0^t + e^{-2t}\left[e^{+u}\right]_0^t = e^{-3t} - \cancel{e}e^{-t} + e^{-3t} + e^{-t} - e^{-2t}$$ 3

$$= 2e^{-3t} - e^{-2t} \quad \text{Q.E.D}$$

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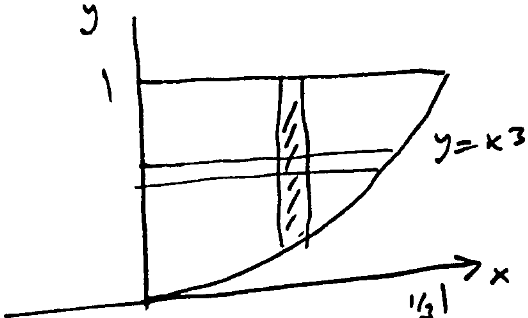
QUESTION

SOLUTION 3

(i)



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$$I = \int_0^1 dy\, e^{y^2} \int_0^{y^{1/3}} x^2 dx = \int_0^1 e^{y^2} dy \left[\frac{x^3}{3}\right]_0^{y^{1/3}}$$

$$= \int_0^1 \frac{y}{3} e^{y^2} dy = \frac{1}{6}\left[e^{y^2}\right]_0^1$$

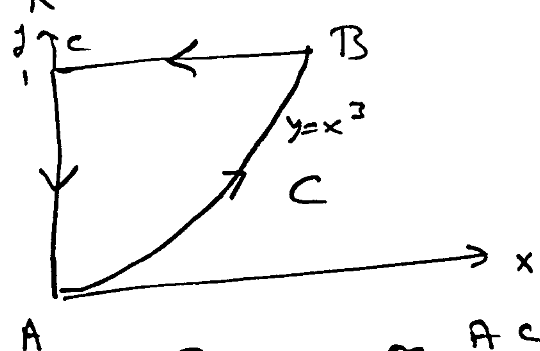
5

$$= \frac{1}{6}\left[e - 1\right]$$

5

(ii) Choose $\phi = \frac{x^3}{3} e^{y^2}$

Then $\iint_R x^2 e^{y^2} dx\,dy = \int_C \frac{x^3}{3} e^{y^2} dy$



C : curve ABC

on AC $x=0$

on BC $dy=0$

5

But on AB $x^3=y$

so $\int_C () = \int_A^B \frac{x^3}{3} e^{y^2} dy$

$$= \int_0^1 \frac{y}{3} e^{y^2} dy = \frac{1}{6}\left[e-1\right]$$ as above

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QUESTION

SOLUTION 4

(i) $\dfrac{1}{(az^2-(a^2+1)z+a)} = \dfrac{1}{(az-1)(z-a)} = \dfrac{1}{(a^2-1)(z-a)} - \dfrac{1}{(a^2-1)(z-1/a)}$

Poles at $z = a,\ 1/a$ (2)

Residues $\dfrac{1}{(a^2-1)}$ at $z = a$ (2)

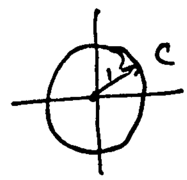
$\dfrac{-1}{(a^2-1)}$ at $z = 1/a$

Integral around arc at ∞ $z = Re^{i\theta}$

$$\int \frac{Rie^{i\theta}\,d\theta}{R^2\left[ae^{2i\theta} - \frac{(a^2+1)e^{i\theta}}{R} + \frac{a}{R^2}\right]} \to 0 \text{ as } R \to \infty$$ (3)

close to the left $I = 0$! Residues cancel out. (3)

close to the right $I = 0$



(ii) on $z = e^{i\theta}$ $dz = iz\,d\theta$ (2)

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + 1/z)$

$$\text{So } J = -i\int_C \frac{dz}{z\left[a(z+1/z) - (a^2+1)\right]}$$

$$= -i\int_C \frac{dz}{(az^2-(a^2+1)z+a)}$$ (4)

$\Rightarrow \boxed{\alpha = a \quad \beta = -(a^2+1) \quad \gamma = a}$

poles at $z = a,\ 1/a$, $a = 2$ $1/a$ is inside unit circle (4)

$J = (-i)\ 2\pi i \operatorname*{Res}_{z=1/2}(\quad)$

$= 2\pi\left\{\dfrac{-1}{3}\right\} = -\dfrac{2\pi}{3}$ ✓

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QUESTION

SOLUTION 5i

i). μ the mean, describes the location of the expected value. σ^2, the variance, describes the amount of dispersion about μ. 2

ii). a).

$$\begin{aligned} P(17 < X < 24) &= P\left(\frac{17-20}{9} < \frac{X-20}{9} < \frac{24-20}{9}\right) \\ &= P\left(-\frac{1}{3} < Z < \frac{4}{9}\right) \\ &= \Phi(4/9) - \Phi(-1/3) \\ &= 0.6700 - (1 - 0.6293) \approx \underline{0.2993} \end{aligned}$$

3

b).

$$\begin{aligned} P(X > 20 | X > 19) &= \frac{P(X > 20 \cap X > 19)}{P(X > 19)} \\ &= \frac{P(X > 20)}{P(X > 19)} \\ &= \frac{P(Z > 0)}{P(Z > -1/9)} \\ &= \frac{1 - \Phi(0)}{1 - (1 - \Phi(1/9))} \\ &= \frac{0.5}{0.5438} \approx \underline{0.9195} \end{aligned}$$

4

iii). $Y \sim N(0, \sigma_1^2 + \sigma_2^2)$ 2

Setter : NA

Setter's signature : NMA

Checker : LVW

Checker's signature : LVW

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QUESTION

SOLUTION 5ii

iv). a).

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \underline{19.375}$$ 1

The sorted values are

4.11, 6.21, 12.28, 13.39, 15.48, 20.42, 21.04, 22.95, 26.01, 27.59, 43.64 1

so the median is $\underline{20.42}$.
The sample standard deviation

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \approx \underline{11.066}$$ 1

b). Use small sample confidence interval

$$\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where we use a t distribution with $n - 1 = 10$ degrees of freedom. Since a 90% confidence interval is required, $1 - \alpha = 0.9$, so $\alpha = 0.1$ and $\alpha/2 = 0.05$. The required value from the t distribution table is 1.8125, so the interval is

$$19.375 \pm 1.8125\left(\frac{11.066}{\sqrt{11}}\right)$$

and so a 90% confidence interval for μ is $(13.328, 25.422)$. 4

c). This claim should be regarded with suspicion. The confidence interval, which has high probability of containing the population mean, does not contain the specific value. 2

Setter : N ADAMS

Setter's signature : N M Adams

Checker : Lynda V. White

Checker's signature :

i).

$$F(t_0) = \int_0^{t_0} \lambda e^{-\lambda t}\,dt$$
$$= \left[\frac{-\lambda e^{-\lambda t}}{\lambda}\right]_0^{t_0}$$
$$= -e^{-\lambda t_0} - (-1)$$
$$= \underline{1 - e^{-\lambda t_0}} \quad t_0 > 0, \qquad F(t_0) = 0 \quad t_0 \le 0$$ 3

ii).

$$P(T > t+s | T > s) = \frac{P(T > t+s \cap T > s)}{P(T > s)}$$
$$= \frac{P(T > t+s)}{P(T > s)}$$
$$= \frac{1 - F(t+s)}{1 - F(s)}$$
$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda(s)}}$$
$$= \underline{e^{-\lambda t}}$$ 4

This is the *memoryless* property of the exponential distribution. 1

iii).

$$E(T) = \int_{-\infty}^{\infty} t f(t)\,dt = \int_0^{\infty} t\lambda e^{-\lambda t}\,dt$$

using integration by parts

$$u = \lambda t \qquad \frac{dv}{dt} = e^{-\lambda t}$$
$$\frac{du}{dt} = \lambda \qquad v = \frac{-e^{-\lambda t}}{\lambda}$$

so,

$$E(X) = -te^{-\lambda t}\Big|_0^{\infty} - \int_0^{\infty} \lambda\left(\frac{-e^{-\lambda t}}{\lambda}\right)\,dt$$
$$= 0 - \frac{e^{-\lambda t}}{\lambda}\Big|_0^{\infty}$$
$$= \underline{\frac{1}{\lambda}}$$ 5

Setter : NA
Checker : LWW
Setter's signature : N M A
Checker's signature : LWW

iv).

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$
$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

taking logs

$$\log(L(\lambda)) = n\log(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

we seek a maximum

$$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

turning points occur when the derivative is 0, so

$$0 = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

so the maximum likelihood estimator is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

and examine the second derivative

$$\frac{d^2\log(L(\lambda))}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$$

to verify that this solution is a maximum.

7

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Checker : LVW

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Checker's signature : LVW