Exam Copy

UNIVERSITY OF LONDON [E2.11 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E2.11

MATHEMATICS

Date Thursday 3rd June 2004 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[*Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.*]

Section A

1. If the Fourier transform of $f(t)$, $-\infty < t < \infty$, is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \,,$$

show that the Fourier transform of $f(-t)$ is $\hat{f}(-\omega)$ and that of $tf(t)$ is $i \dfrac{d}{d\omega} \hat{f}(\omega)$.

If for a positive constant a

$$g(t) = \begin{cases} e^{-at} & , \quad t \geq 0 \,, \\ 0 & , \quad t < 0 \,, \end{cases}$$

$$h(t) = e^{-a|t|} \,, \quad -\infty < t < \infty \,,$$

and

$$k(t) = |t| e^{-a|t|} \,, \quad -\infty < t < \infty \,,$$

find $\hat{g}(\omega)$.

Hence, or otherwise, show that

$$\hat{h}(\omega) = \frac{2a}{a^2 + \omega^2} \quad \text{and} \quad \hat{k}(\omega) = \frac{2(a^2 - \omega^2)}{(a^2 + \omega^2)^2} \,.$$

Find the Fourier Transform of the function $\dfrac{2a}{a^2 + t^2}$.

2. (i) The Laplace transform $\overline{y}(p)$ of a function $y(t)$ is

$$\overline{y}(p) = \int_0^\infty e^{-pt}\, y(t)\, dt \ .$$

Show that, assuming $y(t)$ behaves suitably at infinity, the Laplace transforms of

$$y'(t) \equiv \frac{dy}{dt} \ \text{and} \ y''(t) \equiv \frac{d^2y}{dt^2}$$

are given respectively by

$$\mathcal{L}\{y'\} = p\overline{y}(p) - y(0)$$

and

$$\mathcal{L}\{y''\} = p^2\overline{y}(p) - y'(0) - p\,y(0) \, .$$

Hence solve the differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = g(t)$$

for an arbitrary function $g(t)$, for $t > 0$, given $y(0) = y'(0) = 0$.

(ii) Evaluate $\int_C (P\,dx + Q\,dy)$ anti-clockwise around the boundary of the region R defined by

$$x^2 + y^2 \le a^2, \quad x \ge 0, \quad y \ge 0,$$

taking

$$P = \frac{x^2}{x+y} \qquad \text{and} \qquad Q = -\frac{y^2}{x+y} \, .$$

3. (i) Sketch the region of the xy-plane over which the integral

$$\int_0^1 dx \int_{x^2}^1 4\,x\,e^{y^2}\,dy$$

is taken. Change the order of integration and hence evaluate the integral.

(ii) Sketch the region of the xy-plane over which the integral

$$\int_0^1 dx \int_x^{\sqrt{2x-x^2}} \frac{y}{x^2+y^2} dy$$

is taken.

Use polar co-ordinates to show that the value of the integral is $\frac{1}{2}$.

4. Find all the poles of $\dfrac{1}{z^6+1}$ and the residue at each pole.

For $R > 1$ let C_2 be the upper semi-circular arc of $|z| = R$,

directed from $+R$ to $-R$ and let C_1 be the diameter from $-R$ to R.

If C is the semi-circular path $C_1 + C_2$, calculate

$$\int_C \frac{dz}{z^6+1}\ .$$

By a careful discussion of the limit as $R \to -\infty$ show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}\ .$$

PLEASE TURN OVER

5. A batch of IC chips contains 0.5% defectives. Each IC chip is subjected to a test, which gives a positive result if it identifies a chip as defective. The test correctly identifies a chip as defective with probability 0.99. The test misidentifies as defective 2 in every 100 chips. Let D denote the event that an IC chip is defective and T the event that the test identifies a chip as defective.

   (i) What is the probability that the test is positive when applied to a randomly-selected chip?

   (ii) Given that a randomly-chosen IC chip is declared defective by the tester, compute the probability that it is actually defective.

   Lifetimes (in million of hours) of IC chips are approximately normally distributed with mean μ and variance σ^2. A mainframe manufacturer requires that an IC chip should have a lifetime greater than 4 million hours. He takes a sample of size $n = 100$, with lifetimes denoted by $(x_1, \ldots, x_{100})$, and observes $\sum_{i=1}^{100} x_i = 500$, $\sum_{i=1}^{100} x_i^2 = 2508.9$.

   (iii) Write down expressions for the unbiased estimators for μ and σ^2, and compute the corresponding estimates given the data.

   (iv) Using a large-sample approximation, obtain a 95% confidence interval for μ.

   (v) Using the unbiased estimates of μ and σ^2, evaluate the probability that the lifetime of a randomly selected IC chip is greater than 4 million hours.

   (Approximate your results to 2 decimal points).

PLEASE TURN OVER

6. The random variable X has an Poisson distribution with probability mass function

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x = 0, 1, 2, \ldots$$

where $\lambda > 0$.

(i) Show that $E(X) = \lambda$.

In a manufacturing plant the number of accidents that occurs in a six-month period is a Poisson random variable with mean 2. Assuming that the rate of accidents in any six-month period is constant,

(ii) what is the expected number of accidents per year?

(iii) what is the probability that there will be no accident in a given month?

Suppose we have a random sample, $X_1, \ldots, X_N$, of size N from a Poisson distribution with mean $\mu > 0$.

(iv) Show (and verify) that the maximum likelihood estimator for μ is

$$\widehat{\mu} = \frac{\sum_{i=1}^{N} X_i}{N} = \overline{X}.$$

(v) Suppose that the following data are observed

4, 2, 3, 3, 2, 2, 2, 0, 3, 2 .

Find an estimate of μ.

(vi) Express the expectation of the mean of the random sample

$$\overline{X} = \frac{\sum_{i=1}^{N} X_i}{N}$$

in terms of μ, and write down the formula for the variance of $\overline{X}$.

You may assume that the variance of a Poisson random variable is equal to its mean μ.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary},\ |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x=a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y=y(x)$, then $f=F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x=x(t)$, $y=y(t)$, then $f=F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x=x(u,\,v)$, $y=y(u,\,v)$, then $f=F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u},\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x=0,\ f_y=0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b)<0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b)>0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2\ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. For the **sample space**, Ω, the impossible event $\emptyset$, and events A, B, C:

$$P(\Omega) = 1, \quad P(\emptyset) = 0, \quad P(\overline{A}) = 1 - P(A).$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
$$P(A \cup B \cup C)$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

Conditional probability: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$.

The **odds** in favour of A is the ratio $P(A)/P(\overline{A})$.

Multiplication rule: $P(A \cap B) = P(A \mid B)\, P(B)$.

Chain rule: $P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$.

Bayes' rule: $$P(A \mid B) = \frac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$$

Independence: Events A and B are **independent** if $P(B \mid A) = P(B)$.

Events A, B, C are **independent** if $P(A \cap B \cap C) = P(A)P(B)P(C)$,

$$P(A \cap B) = P(A)P(B), \ P(B \cap C) = P(B)P(C), \ P(C \cap A) = P(C)P(A).$$

2. A discrete random variable X has the **probability mass function** $\{p_x\} = \{P(X = x)\}$

The **expectation:** $E(X) = \mu = \sum_x x p_x$.

From random sample $x_1, \ldots, x_n$, the **sample mean** $\overline{x} = (1/n) \sum_k x_k$ estimates $E(X)$.

The **variance:** $\text{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \{E(X)\}^2$, where $E(X^2) = \sum_x x^2 p_x$.

The **sample variance:** $s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates $\text{var}(X)$.

The **standard deviation:** $\text{sd}(X) = \sigma = \sqrt{\text{var}(X)}$.

For grouped data: if the value y is observed with frequency n_y, then

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y.$$

Estimated **skewness** is $\dfrac{1}{n-1} \sum_k \left(\dfrac{x_k - \overline{x}}{s} \right)^3$, estimated **kurtosis** is $\dfrac{1}{n-1} \sum_k \left(\dfrac{x_k - \overline{x}}{s} \right)^4$

3. **Binomial distribution:** X is *Binomial*(n, θ).

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n); \qquad \mu = n\theta, \ \sigma^2 = n\theta(1-\theta).$$

Poisson distribution: X is *Poisson*(λ).

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \ \ (\text{with } \lambda > 0); \quad \mu = \lambda, \ \sigma^2 = \lambda.$$

Geometric distribution: X is *Geometric*(θ).

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots); \quad \mu = \frac{1}{\theta}, \ \sigma^2 = \frac{1-\theta}{\theta^2}.$$

4. For **continuous** random variables, the **cumulative distribution function** (cdf)

$$F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$$

The **probability density function** (pdf) $\quad f(x) \; = \; \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,, \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x\,, \quad \mathrm{var}\,(X) = E(X^2) - \{E(X)\}^2\,.$$

5. **Uniform distribution:** X is *Uniform*(α, β).

$$f(x) = \begin{cases} 1/(\beta-\alpha) & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \begin{aligned} &\mu = (\alpha+\beta)/2, \\ &\sigma^2 = (\beta-\alpha)^2/12. \end{aligned}$$

Exponential distribution: X is *Exponential*(λ).

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \begin{aligned} &\mu = 1/\lambda, \\ &\sigma^2 = 1/\lambda^2. \end{aligned}$$

Gamma distribution: X is *Gamma*(ν, λ).

$$f(x) = \begin{cases} \{1/\Gamma(\nu)\}\, \lambda^\nu\, x^{\nu-1}\, e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \begin{aligned} &\mu = \nu/\lambda, \\ &\sigma^2 = \nu/\lambda^2. \end{aligned}$$

Normal distribution: X is $N(\mu, \sigma^2)$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty); \quad E(X) = \mu, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution: Y is $N(0,1)$.

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$.

For Y we write $\phi(y)$ for the pdf $f(y)$ and $\Phi(y)$ for the cdf $F(y)$

6. The lifetime T of a device in continuous operation with pdf $f(t)$ $(t > 0)$:

The **reliability** at time t: $R(t) \; = \; P(T > t)$.

The **failure rate** or **hazard rate**: $h(t) \; = \; f(t)/R(t)$.

The **hazard function**: $H(t) \; = \; \int_0^t h(t_0)\,\mathrm{d}t_0 \; = \; -\ln\{R(t)\}$

The **Weibull distribution** *Weibull*(α, β) has $H(t) = \beta t^\alpha$.

For a system of k devices, which operate independently:

The **system reliability**, R, is the probability of a path of operating devices.

Let $R_i = P(D_i) = P($"device i operates"$)$.

A system of devices in **series** fails if any device fails.

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k\,.$$

A system of devices in **parallel** operates if any device operates.

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)\,.$$

7. The **covariance** of X and Y:

$$\text{cov}(X,Y) = E\{(X - E(X))(Y - E(Y))\} = E(XY) - E(X)E(Y)$$

The estimate of cov (X, Y) from n pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ is

$$s_{xy} = \frac{1}{n-1} S_{xy} \quad \text{where} \quad S_{xy} = \textstyle\sum_k x_k y_k - \frac{1}{n} (\sum_i x_i) \left(\sum_j y_j\right)$$

The **correlation coefficient**: $\rho = \text{corr}(X,Y) = \dfrac{\text{cov}(X,Y)}{\text{sd}(X) \cdot \text{sd}(Y)}$

The **sample correlation coefficient**: $r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ,

where $S_{xx} = (n-1)s_{xx}$, $S_{yy} = (n-1)s_{yy}$, and s_{xx} and s_{yy} are s^2 calculated from the xs and ys respectively.

If X and Y have the joint pdf $f(x, y)$:
the **marginal pdf** for X is $f_X(x) = \displaystyle\int_{-\infty}^{\infty} f(x, y_0)\, dy_0$

the **conditional pdf** for X given $Y = y$ is $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ provided $f_Y(y) > 0$

The pdf for $Z = X + Y$ is $f_Z(z) = \displaystyle\int_{x=-\infty}^{\infty} f_X(x)\, f_{Y|X}(z - x \,|\, x) dx$

$$E(X+Y) = E(X) + E(Y), \quad \text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and cov $(X, Y) = c$, then $X+Y$ is $N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2 + 2c)$

8. **Chi-squared distribution** χ_k^2: $E(Z) = k$, var $(Z) = 2k$

If $Y_1, \ldots, Y_k$ are independent $N(0,1)$ then $Z = Y_1^2 + \cdots + Y_k^2$ is χ_k^2.

For a random sample from $N(\mu, \sigma^2)$, $(n-1)s^2/\sigma^2$ is from χ_{n-1}^2, and
$\sqrt{n}(\bar{x} - \mu)/s$ is from t_{n-1}, the **Student t distribution** on $n - 1$ degrees of freedom

9. If t estimates θ, the **standard error** of t, se(t), is sd(T), the standard deviation of the **sampling distribution** of t, and **bias**$(t) = E(T - \theta)$.

The **mean square error**: $E\{(T-\theta)^2\} = \text{var}(T) + \{\text{bias}(t)\}^2$.

If $\bar{x}$ estimates μ, then bias$(\bar{x}) = 0$, se$(\bar{x}) = \sigma/\sqrt{n}$, $\widehat{\text{se}}(\bar{x}) = s/\sqrt{n}$, and MSE $= \sigma^2/n$.

The **likelihood** is the joint probability as a function of the unknown parameter θ.

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = P(X_1 = x_1) \cdots P(X_n = x_n) \quad \text{(discrete distribution)}$$
$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = f(x_1) f(x_2) \cdots f(x_n) \quad \text{(continuous distribution)}$$

The **maximum likelihood estimator** (MLE) is $\hat{\theta}$ for which the likelihood is a maximum.

10. If t estimates θ, a 95% **confidence interval** for θ is an estimated interval that contains 95% of the sampling distribution of θ.

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% CI for μ is $(\overline{x} - 1.96\sigma/\sqrt{n},\ \overline{x} + 1.96\sigma/\sqrt{n})$.
If σ^2 is estimated, then from the table of t_{n-1} we find $t_0 = t_{n-1,0.05}$. Then the 95% CI for μ is $(\overline{x} - t_0 s/\sqrt{n},\ \overline{x} + t_0 s/\sqrt{n})$.

A **significance test** of H_0 rejects H_0 if, assuming that H_0 is true, a test statistic is in a rejection region of its sampling distribution.

The **chi-squared goodness-of-fit test** checks how well a fitted distribution fits the data:
The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum(n_y - \widehat{n}_y)^2/\widehat{n}_y$ is referred to the table of χ^2_k with significance point p, where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ.

11. To fit the **linear regression** model $y = \alpha + \beta x$ by $\widehat{y} = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}$, $\widehat{\beta} = S_{xy}/S_{xx}$.

The **residual sum of squares**, $\text{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \text{ estimates } \sigma^2; \qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2} \text{ is } \chi^2_{n-2}$$

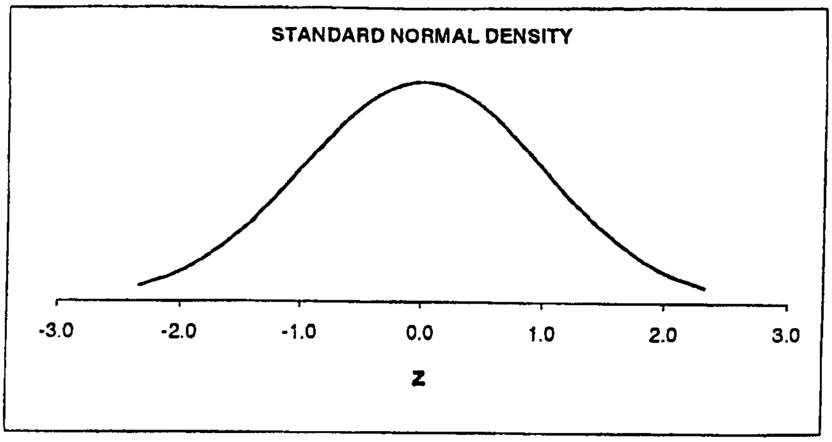
The predictor $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ of y when $X = x$ is $\widehat{\text{var}}\,(\widehat{y}_x) = \left\{\dfrac{1}{n} + \dfrac{(x-\overline{x})^2}{S_{xx}}\right\}\widehat{\sigma^2}$

$$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}\,(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}\,(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}\,(\widehat{y}_x)} \quad \text{are each } t_{n-2}.$$

A future single observation y' at $X = x$ has prediction error $\widehat{y}_x - y'$ which has variance $\text{var}\,(\widehat{y}_x) + \sigma^2$

The 95% **prediction interval** for y' at $X = x$ is $\widehat{y}_x \pm t_{n-2,0.05}\sqrt{\{\widehat{\text{var}}\,(\widehat{y}_x) + \widehat{\sigma^2}\}}$

THE STANDARD NORMAL DISTRIBUTION FUNCTION



Entries in table are probabilities p such that Φ(z)=p

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

TABLE OF THE STANDARD NORMAL CDF

Entries in table are ordinates x such that F(x)=p where F(.) is the Student cdf

D. of F.	0.8	0.9	0.95	0.975	0.99	0.995	0.999
				p			
1	1.3764	3.0777	6.3137	12.7062	31.8210	63.6559	318.2888
2	1.0607	1.8856	2.9200	4.3027	6.9645	9.9250	22.3285
3	0.9785	1.6377	2.3534	3.1824	4.5407	5.8408	10.2143
4	0.9410	1.5332	2.1318	2.7765	3.7469	4.6041	7.1729
5	0.9195	1.4759	2.0150	2.5706	3.3649	4.0321	5.8935
6	0.9057	1.4398	1.9432	2.4469	3.1427	3.7074	5.2075
7	0.8960	1.4149	1.8946	2.3646	2.9979	3.4995	4.7853
8	0.8889	1.3968	1.8595	2.3060	2.8965	3.3554	4.5008
9	0.8834	1.3830	1.8331	2.2622	2.8214	3.2498	4.2969
10	0.8791	1.3722	1.8125	2.2281	2.7638	3.1693	4.1437
11	0.8755	1.3634	1.7959	2.2010	2.7181	3.1058	4.0248
12	0.8726	1.3562	1.7823	2.1788	2.6810	3.0545	3.9296
13	0.8702	1.3502	1.7709	2.1604	2.6503	3.0123	3.8520
14	0.8681	1.3450	1.7613	2.1448	2.6245	2.9768	3.7874
15	0.8662	1.3406	1.7531	2.1315	2.6025	2.9467	3.7329
16	0.8647	1.3368	1.7459	2.1199	2.5835	2.9208	3.6861
17	0.8633	1.3334	1.7396	2.1098	2.5669	2.8982	3.6458
18	0.8620	1.3304	1.7341	2.1009	2.5524	2.8784	3.6105
19	0.8610	1.3277	1.7291	2.0930	2.5395	2.8609	3.5793
20	0.8600	1.3253	1.7247	2.0860	2.5280	2.8453	3.5518
21	0.8591	1.3232	1.7207	2.0796	2.5176	2.8314	3.5271
22	0.8583	1.3212	1.7171	2.0739	2.5083	2.8188	3.5050
23	0.8575	1.3195	1.7139	2.0687	2.4999	2.8073	3.4850
24	0.8569	1.3178	1.7109	2.0639	2.4922	2.7970	3.4668
25	0.8562	1.3163	1.7081	2.0595	2.4851	2.7874	3.4502
26	0.8557	1.3150	1.7056	2.0555	2.4786	2.7787	3.4350
27	0.8551	1.3137	1.7033	2.0518	2.4727	2.7707	3.4210
28	0.8546	1.3125	1.7011	2.0484	2.4671	2.7633	3.4082
29	0.8542	1.3114	1.6991	2.0452	2.4620	2.7564	3.3963
30	0.8538	1.3104	1.6973	2.0423	2.4573	2.7500	3.3852
31	0.8534	1.3095	1.6955	2.0395	2.4528	2.7440	3.3749
32	0.8530	1.3086	1.6939	2.0369	2.4487	2.7385	3.3653
33	0.8526	1.3077	1.6924	2.0345	2.4448	2.7333	3.3563
34	0.8523	1.3070	1.6909	2.0322	2.4411	2.7284	3.3480
35	0.8520	1.3062	1.6896	2.0301	2.4377	2.7238	3.3400
36	0.8517	1.3055	1.6883	2.0281	2.4345	2.7195	3.3326
37	0.8514	1.3049	1.6871	2.0262	2.4314	2.7154	3.3256
38	0.8512	1.3042	1.6860	2.0244	2.4286	2.7116	3.3190
39	0.8509	1.3036	1.6849	2.0227	2.4258	2.7079	3.3127
40	0.8507	1.3031	1.6839	2.0211	2.4233	2.7045	3.3069
41	0.8505	1.3025	1.6829	2.0195	2.4208	2.7012	3.3012
42	0.8503	1.3020	1.6820	2.0181	2.4185	2.6981	3.2959
43	0.8501	1.3016	1.6811	2.0167	2.4163	2.6951	3.2909
44	0.8499	1.3011	1.6802	2.0154	2.4141	2.6923	3.2861
45	0.8497	1.3007	1.6794	2.0141	2.4121	2.6896	3.2815
46	0.8495	1.3002	1.6787	2.0129	2.4102	2.6870	3.2771
47	0.8493	1.2998	1.6779	2.0117	2.4083	2.6846	3.2729
48	0.8492	1.2994	1.6772	2.0106	2.4066	2.6822	3.2689
49	0.8490	1.2991	1.6766	2.0096	2.4049	2.6800	3.2651
50	0.8489	1.2987	1.6759	2.0086	2.4033	2.6778	3.2614
55	0.8482	1.2971	1.6730	2.0040	2.3961	2.6682	3.2451
60	0.8477	1.2958	1.6706	2.0003	2.3901	2.6603	3.2317
65	0.8472	1.2947	1.6686	1.9971	2.3851	2.6536	3.2204
70	0.8468	1.2938	1.6669	1.9944	2.3808	2.6479	3.2108
75	0.8464	1.2929	1.6654	1.9921	2.3771	2.6430	3.2024
80	0.8461	1.2922	1.6641	1.9901	2.3739	2.6387	3.1952
85	0.8459	1.2916	1.6630	1.9883	2.3710	2.6349	3.1889
90	0.8456	1.2910	1.6620	1.9867	2.3685	2.6316	3.1832
95	0.8454	1.2905	1.6611	1.9852	2.3662	2.6286	3.1783
100	0.8452	1.2901	1.6602	1.9840	2.3642	2.6259	3.1738
150	0.8440	1.2872	1.6551	1.9759	2.3515	2.6090	3.1455
200	0.8434	1.2858	1.6525	1.9719	2.3451	2.6006	3.1315
250	0.8431	1.2849	1.6510	1.9695	2.3414	2.5956	3.1231
∞	0.8416	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

Tables of the Student-t distribution

ISE
2nd yr Maths
2004
(E2.11)

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

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PAPER ISE 20(

QUESTION

SOLUTION 1

Given $\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \hat{f}(\omega)$ ① then F.T of $f(-t)$ is

$$\int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt \overset{t=-s}{=} \int_{+\infty}^{-\infty} f(s)e^{i\omega s}(-ds)$$

$$= \int_{-\infty}^{\infty} f(s)e^{i\omega s}\,ds = \hat{f}(-\omega)$$ (2)

Differentiate ① w.r.t ω

$$\int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial\omega}(e^{-i\omega t})\,dt = \frac{d\hat{f}}{d\omega} = \int_{-\infty}^{\infty} -it\,f(t)e^{-i\omega t}\,dt$$

$\therefore$ F.T. of $t f(t)$ is $i\dfrac{d\hat{f}(\omega)}{d\omega}$ (5)

$$\hat{g}(\omega) = \int_0^{\infty} e^{-at}e^{-i\omega t}\,dt = \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} = \frac{1}{a+i\omega} \quad \text{Since } a>0$$ (3)

$h(t) = g(t) + g(-t)$ so

$$\hat{h}(\omega) = \hat{g}(\omega) + \hat{g}(-\omega) = \frac{1}{a+i\omega} + \frac{1}{a-i\omega} = \frac{2a}{a^2+\omega^2}$$ (4)

$t g(t)$ has F.T $i\dfrac{d}{d\omega}\left(\dfrac{1}{a+i\omega}\right) = \dfrac{-i^2}{(a+i\omega)^2} = \dfrac{1}{(a+i\omega)^2}$

$k(t) = t g(t) + (-t g(-t))$ has F.T.

$$\frac{1}{(a+i\omega)^2} + \frac{1}{(a-i\omega)^2} = \frac{2(a^2-\omega^2)}{(a^2+\omega^2)^2}.$$ (3)

Then use symmetry formula

$\hat{f}(t)$ has F.T. $2\pi f(-\omega)$

So F.T. of $\dfrac{2a}{a^2+t^2}$ is $2\pi h(-\omega) = 2\pi e^{-a|\omega|}$ (3)

Setter: J.R. CASH   Setter's signature: JR Cash

Checker: C J RIDLER ROWE   Checker's signature: CJRR

PAPER ISE 2.6

QUESTION

SOLUTION 2

F T of $\frac{dy}{dt}$ is $\int_0^\infty e^{-pt}\frac{dy}{dt}\,dt$

$$= \left[e^{-pt}y\right]_0^\infty + \int_0^\infty p e^{-pt} y\,dt$$

$$= -y(0) + p\bar{y}(p)$$ (2)

Similarly F.T. of $\frac{d^2y}{dt^2}$ is $\int_0^\infty e^{-pt}\frac{d^2y}{dt^2}\,dt$

$$= \left[e^{-pt}\frac{dy}{dt}\right]_0^\infty + \int_0^\infty p e^{-pt}\frac{dy}{dt}\,dt$$

$$= -y'(0) + p\int_0^\infty e^{-pt}\frac{dy}{dt}\,dt$$ (2)

$$= -y'(0) - p(y(0)) + p^2\bar{y}(p)$$

Consider $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = g(t)$ and take L.T.

$$p^2\bar{y}(p) - y'(0) - py(0) - 3\left(p\bar{y}(p) - y(0)\right) + 2\bar{y}(p) = \bar{g}$$ (2)

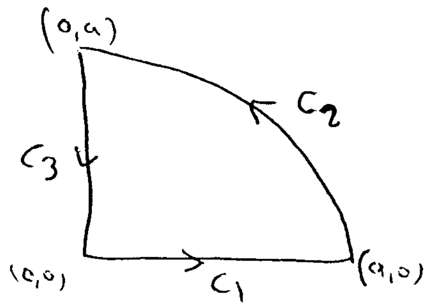
i.e. $p^2\bar{y} - 3p\bar{y} + 2\bar{y} = \bar{g}$

So $\bar{y} = \frac{\bar{g}}{p^2-3p+2} = \frac{\bar{g}}{(p-1)(p-2)} = \left(\frac{1}{p-2} - \frac{1}{p-1}\right)\bar{g}$ (2)

$$\therefore y(t) = \int_0^t \left[e^{2(t-u)} - e^{(t-u)}\right]g(u)\,dy$$

$$= e^{2t}\int_0^t e^{-2u}g(u)\,du - e^{t}\int_0^t e^{-u}g(u)\,dy.$$ (2)

2)



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PAPER ISE 2.6

QUESTION

SOLUTION 2 ctd

$$I = \int_C P\,dx + Q\,dy$$

On C_1 $\quad x = t,\ y = 0 \qquad I = \int_C P\,dx + Q\,dy = \int_{t=0}^{a}\left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt$

$$= \int_0^a t\cdot 1\,dt = a^2/2$$ (2)

On C_3 $\quad I = -\int_{C_3} P\,dx + Q\,dy \qquad x = 0,\ y = t$

$$= -\int_{t=0}^{a}\left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt = -\int_0^a -t\cdot 1\,dt = a^2/2$$ (2)

On C_2 put $x = a\cos\theta,\ y = a\sin\theta$ (1)

$$I_2 = \int_{\theta=0}^{\theta=\pi/2}\left(P\frac{dx}{d\theta} + Q\frac{dy}{d\theta}\right)d\theta$$

$$= \int_0^{\pi/2} \frac{a^2\cos^2\theta(-a\sin\theta) - a^2\sin^2\theta(a\cos\theta)}{a\cos\theta + a\sin\theta}\,d\theta$$

$$= \int_0^{\pi/2} -a^2\sin\theta\cos\theta\,d\theta = \left[-\frac{a^2}{2}\sin^2\theta\right]_0^{\pi/2} = -\frac{a^2}{2}$$ (3)

$$\therefore I = I_1 + I_2 + I_3 = a^2/2$$

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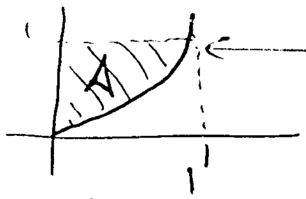
PAPER ISE 2.6

QUESTION

SOLUTION 3

(i)



$y = x^2, \quad x = \sqrt{y}$ (3)

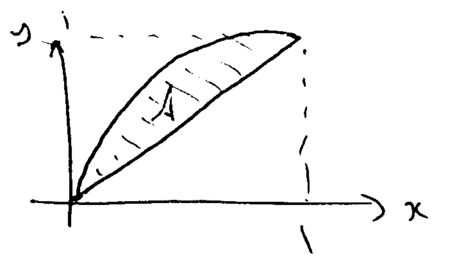
$$I = \int_0^1 dx \int_{x^2}^1 4x e^{y^2}\, dy = \iint_A 4x e^{y^2}\, dx\, dy$$ (2)

$$= \int_0^1 dy \int_0^{\sqrt{y}} 4x e^{y^2}\, dx$$ (2)

$$= \int_0^1 \left[2x^2 e^{y^2}\right]_{x=0}^{\sqrt{y}} dy$$ (1)

$$= \int_0^1 2y e^{y^2}\, dy = \left[e^{y^2}\right]_0^1 = e - 1.$$ (1)

(ii)



$y = \sqrt{2x - x^2}$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$ Circle (3)

Let $x = r\cos\theta$, $y = r\sin\theta$ then $x^2 - 2x + y^2 = 0 \Rightarrow r^2 - 2r\cos\theta = 0$

so $r = 0$ or $r = 2\cos\theta$ (2)

$$\therefore I = \iint_A \frac{r\sin\theta}{r^2}\, r\, dr\, d\theta$$ (3)

$$= \int_{\pi/4}^{\pi/2} d\theta \int_0^{2\cos\theta} \sin\theta\, dr$$ (1)

$$= \int_{\pi/4}^{\pi/2} 2\sin\theta\cos\theta\, d\theta = \int_{\pi/4}^{\pi/2} \sin 2\theta\, d\theta$$

$$= \left[-\tfrac{1}{2}\cos 2\theta\right]_{\pi/4}^{\pi/2} = 1/2$$ (2)

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Checker :   Checker's signature :

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

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PAPER ISE 2.6

QUESTION

SOLUTION 4

Poles where $z^6 = -1 = e^{i\pi}$ ∴ $z = e^{i(\pi/6 + n\pi/3)} = e^{i\theta}$

where $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$.

If we take η as one of these values (so $\eta^6 = -1$) then the residue of $\frac{1}{z^6+1}$ at η is $\frac{1}{\frac{d}{dz}(z^6+1)} = \frac{1}{6z^5} = -\frac{\eta}{6}$

So residues

$e^{i\pi/6}$ $\quad -\frac{1}{6}\left(\frac{\sqrt{3}}{2} + i/2\right)$

$e^{i\pi/2}$ $\quad -i/6$

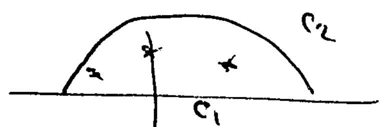
$e^{5\pi i/6}$ $\quad -1/6\left(-\frac{\sqrt{3}}{2} + i/2\right)$

$e^{-i\pi/6}$ $\quad 1/6\left(\frac{\sqrt{3}}{2} + i/2\right)$

$-e^{i\pi/2}$ $\quad i/6$

$-e^{5\pi i/6}$ $\quad 1/6\left(-\frac{\sqrt{3}}{2} + i/2\right)$ (10)

$$\int_{C=C_1+C_2} \frac{dz}{z^6+1} = 2\pi i\left(\text{Res at } e^{i\pi/6} + \text{Res at } i + \text{Res at } e^{2\pi i/3}\right)$$



$$= 2\pi i\left\{-\frac{\sqrt{3}}{2} - \frac{i}{2} - i + \frac{\sqrt{3}}{2} - i/2\right\}/6 = \frac{2\pi}{3}$$ (5)

by residue thm

$$\int_{C_1} \frac{dz}{z^6+1} = \int_{-R}^{R} \frac{dx}{x^6+1}$$ On C_2 $|z^6+1| \geq R^6 - 1$ so

$$\left|\int_{C_2} \frac{dz}{z^6+1}\right| \leq \frac{\pi R}{R^6-1} \to 0 \text{ as } R \to \infty$$

∴ For any $R > 1$ $\quad \frac{2\pi}{3} = \int_{C_1} + \int_{C_2}$ and as $R \to \infty$ (5)

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

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4

Problem 1
(a)

$$P(D) = \frac{0.5}{100} = 0.005$$

$$P(\bar{D}) = 1 - P(D) = 0.995$$

$$P(T \mid D) = 0.99$$

$$P(T \mid \bar{D}) = \frac{2}{100} = 0.02$$

Apply Law of Total Probabilities

$$P(T) = P(T \mid D)P(D) + P(T \mid \bar{D})P(\bar{D})$$

$$= (0.99 \times 0.005) + (0.02 \times 0.995)$$

$$= 0.00495 + 0.0199 = 0.02485$$

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(b) Bayes Theorem 4

$$P(D|T) = \frac{P(D)\,P(T|D)}{P(T)}$$

$$= \frac{0.005 \times 0.99}{0.02485} = \frac{0.00495}{0.02485} =$$

$$\simeq 0.1992$$

(c) $\hat{\mu} = \dfrac{\sum_{i=1}^{m} x_i}{m} = \bar{x}$ 5

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

$$\hat{\mu} = \frac{500}{100} = 5$$

MATHEMATICS FOR ENGINEERING STUDENTS

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PAPER

QUESTION

SOLUTION 15

$$\hat{\sigma}^2 = \frac{1}{99}\left[\Sigma(x_i^2) + n\bar{x}^2 - 2n\bar{x}^2\right]$$

$$= \frac{1}{99}\left(\Sigma x_i^2 - 100\bar{x}^2\right) =$$

$$= \frac{8.91}{99} = 0.09$$

(d) 4

95% confidence interval

$$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$z_{\alpha/2} = 1.96 \quad , \quad n = 100$$

$$\hat{\sigma} = \sqrt{0.09} = 0.3$$

$$\bar{X} \pm z_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} = 5 \pm 1.96\,(0.03)$$

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C. I. is

$$(4.9412,\ 5.0588)$$

(i) 3

$$X \sim N(5, 0.09)$$

$$P(X > 4) = 1 - P(X \le 4)$$

$$= 1 - P\left(\frac{X-5}{0.3} \le \frac{4-5}{0.3}\right)$$

$$= 1 - P\left(Z \le \frac{-1}{0.3}\right)$$

$$= 1 - \left[1 - P\left(Z \le \frac{1}{0.3}\right)\right]$$

$$= P\left(Z \le \frac{1}{0.3}\right) \simeq P(Z \le 3.333)$$

$$\simeq 0.9996$$

$$Z \sim N(0,1)$$

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PAPER

QUESTION

SOLUTION 26

PROBLEM 2

(a) 5

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{\lambda^{x} e^{-\lambda}}{(x-1)!} =$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} e^{\lambda} \lambda = \lambda$$

(b) 2

λ = mean in a 6 months period

= 2

μ = mean in a year period

= 2 × 2 = 4

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(e) 3

μ = mean in a month period

$$= \frac{2}{6} = \frac{1}{3}$$

$$P(X=0) = \frac{\mu^0 e^{-\mu}}{0!} = e^{-1/3} \simeq 0.7165$$

(d) 5

Likelihood

$$L(\mu) = \prod_{i=1}^{n} \frac{e^{-\mu} \mu^{x_i}}{x_i!}$$

$$= \frac{e^{-\mu n} \mu^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}.$$

$$\log(L(\mu)) = -\mu n + (\Sigma x_i) \log \mu - \log(\Pi x_i!)$$

MATHEMATICS FOR ENGINEERING STUDENTS

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PAPER ISE 2

QUESTION

SOLUTION 26

$$\frac{d \log(L(\mu))}{d\mu} = -m + \frac{\sum x_i}{\mu} = 0$$

$$\hat{\mu} = \frac{\sum x_i}{m} = \bar{x}$$

$$\frac{d^2 \log(L(\mu))}{d\mu^2} = -\frac{\sum x_i}{\mu^2} < 0$$

$\Rightarrow$ maximum

(e)

$$\hat{\mu} = \frac{23}{10} = 2.3$$ 1

(f) 4

$$E(\bar{X}) = \frac{\sum_{i=1}^{m} \mu}{m} = \mu$$

$$V(\bar{X}) = \frac{1}{m}\mu$$

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