E4.45
C5.21
SO22
ISE4.47

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE/ISE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Corrected Copy

Thursday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : P.A. Naylor

Special Information for the Invigilators: NONE

Information for Candidates:

1. *Sub-sampling by an integer N:*

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

2. *Poisson Summation formula:*

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k)e^{j2\pi kt}.$$

3. *Geometric Series:*

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho} \qquad |\rho| < 1.$$

4. *Admissible Scaling Function:*
A function $\varphi(t)$ is an admissible scaling function of $L_2(\mathbb{R})$ if and only if it satisfies the three following conditions:

(a) Riesz basis criterion: there exists two constants $A > 0$ and $B < +\infty$ such that

$$A \leq \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 \leq B$$

(b) Two scale relation

$$\varphi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0[k]\varphi(2t-k)$$

(c) Partition of unity

$$\sum_{k\in\mathbb{Z}} \varphi(t-k) = 1.$$

The Questions

1. Consider the four-channel filter bank shown in Figure 1

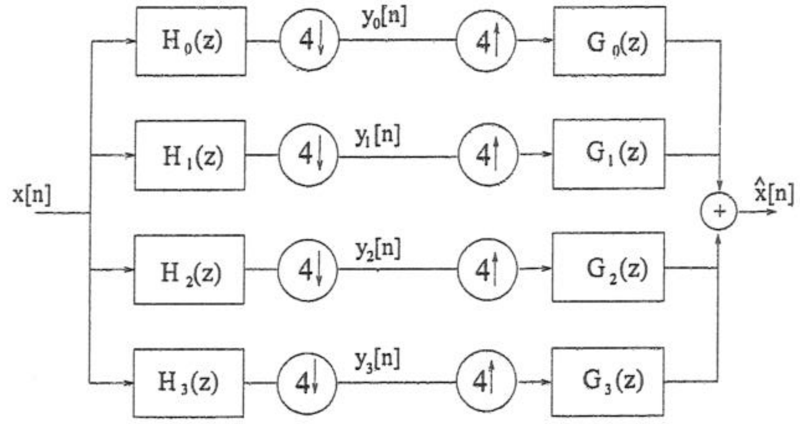

Figure 1: Four-channel filter bank.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the four perfect reconstruction conditions the filters have to satisfy.

[8]

(b) Assume that $G_0(z) = \frac{1}{2}(1+z^{-1}+z^{-2}+z^{-3})$, and $G_1(z) = \frac{1}{2}(1+z^{-1}-z^{-2}-z^{-3})$, design two four-taps filters $G_2(z)$ and $G_3(z)$ such that the following conditions are satisfied:

$$\langle g_i[n], g_j[n-4k]\rangle = \delta_{i,j}\cdot\delta_k \quad i,j = 0,1,2,3 \text{ and } k \in \mathbb{Z}.$$

[6]

(c) Given the synthesis filters $g_i[n]$ of part (b), choose $H_i(z) = G_i(z^{-1})$, for $i = 0,1,2,3$. Now, the filter bank is iterated on the H_0 branch to form a 2-level decomposition. Draw either the synthesis or the analysis filter bank of the equivalent 7-channel filter bank clearly specifying all the transfer functions and downsampling factors.

[6]

2. Consider the two-channel filter bank of Figure 2.

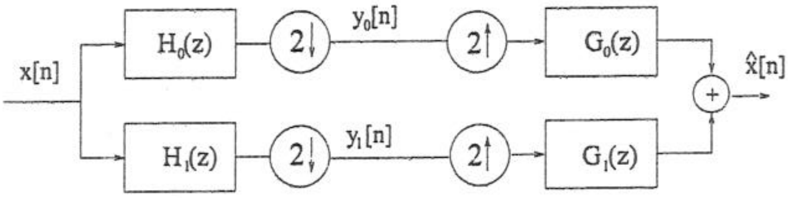

Figure 2: Two-channel filter bank.

(a) Assume that $G_0(z) = \frac{1}{2\sqrt{2}}(1+z^{-1})(1+z)$ and assume that $H_0(z) = (1+z)(1+z^{-1})B(z)$. Determine the shortest symmetric polynomial $B(z)$ such that $P(z) + P(-z) = 2$, where $P(z) = H_0(z)G_0(z)$.

[8]

(b) Given the filters $G_0(z)$ and $H_0(z)$ of part (a), design the filters $H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction biorthogonal filter bank.

[6]

(c) Based on the polynomial $P(z)$ of part (a), construct an orthogonal filter bank.

[6]

3. Assume that two functions $\varphi_0(t)$ and $\varphi_1(t)$ are valid scaling functions. Show that the function $\varphi_2(t) = \varphi_0(t) * \varphi_1(t)$ given by the convolution of $\varphi_0(t)$ with $\varphi_1(t)$ satisfies:

   (a) The partition of unity:

   $$\sum_{n=-\infty}^{\infty} \varphi_2(t-n) = 1$$

   (Hint: Use Poisson sum formula).

   [7]

   (b) The two-scale equation:

   $$\varphi_2(t) = \sqrt{2} \sum_n g_2[n] \varphi_2(2t-n).$$

   [7]

   (c) Now assume that $\varphi_0(t) = \beta_0(t)$ and $\varphi_1(t) = \beta_1(t)$, where $\beta_0(t)$ is the box function with Fourier transform $\hat{\beta}_0(\omega) = \frac{1-e^{j\omega}}{j\omega}$ and $\beta_1(t) = \beta_0(t) * \beta_0(t)$. Thus, $\varphi_2(t) = \beta_0(t) * \beta_1(t)$. Find the exact expression of the filter $g_2[n]$ that leads to the two-scale equation:

   $$\varphi_2(t) = \sqrt{2} \sum_n g_2[n] \varphi_2(2t-n).$$

   [6]

4. Consider the system shown in Figure 3

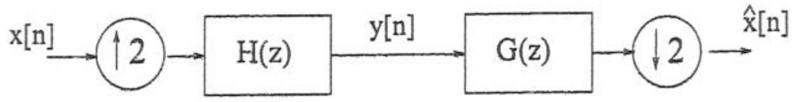

Figure 3: An interpolator.

(a) What does the product filter $P(z) = H(z)G(z)$ have to satisfy in order to have perfect reconstruction such that $\hat{x}[n] = x[n]$?

[4]

(b) Assume that $H(z) = (z^{-2} + z^{-1} + 1 + z + z^2)$. Find the shortest symmetric filter $G(z)$ such that perfect reconstruction is achieved.

[4]

(c) Now assume that $H(z) = (1 + z + z^2 + z^3)$. Design $G(z)$ so that the output $\hat{X}(z) = 0$.

[4]

(d) *Infinite products and Haar scaling function*

i. Consider the following product:

$$p_i = \prod_{k=0}^{i} a^{b^k} \quad |b| < 1,$$

show that $\lim_{i\to\infty} p_i = a^{1/(1-b)}$.

[4]

ii. Assume that $M_0(\omega) = G_0(e^{j\omega})/\sqrt{2}$ where $G_0(e^{j\omega}) = (1+e^{-j\omega})/\sqrt{2}$ is the Haar low-pass filter. Show that

$$\lim_{i\to\infty} \prod_{k=1}^{i} M_0(\omega/2^k) = e^{-j\omega/2}\frac{\sin(\omega/2)}{\omega/2}.$$

Hint: Use the identity $\cos(\omega) = \sin(2\omega)/2\sin(\omega)$.

[4]

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QUESTION 1 Wavelets + Applications 2007

$\frac{1}{10}$

(a)
$$
\begin{aligned}
\hat{X}(z) = \frac{1}{4}\Big[& G_0(z)\Big(X(z)H_0(z) + X(W_4^1 z)H_0(W_4^1 z) + \\
& + X(W_4^2 z)H_0(W_4^2 z) + X(W_4^3 z)H_0(W_4^3 z)\Big) \\
& + G_1(z)\Big(X(z)H_1(z) + X(W_4^1 z)H_1(W_4^1 z) + \\
& + X(W_4^2 z)H_1(W_4^2 z) + X(W_4^3 z)H_1(W_4^3 z)\Big) \\
& + G_2(z)\Big(X(z)H_2(z) + X(W_4^1 z)H_2(W_4^1 z) + \\
& + X(W_4^2 z)H_2(W_4^2 z) + X(W_4^3 z)H_2(W_4^3 z)\Big) \\
& + G_3(z)\Big(X(z)H_3(z) + X(W_4^1 z)H_3(W_4^1 z) + \\
& + X(W_4^2 z)H_3(W_4^2 z) + X(W_4^3 z)H_3(W_4^3 z)\Big)\Big].
\end{aligned}
$$

PR CONDITIONS

$$G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z) + G_3(z)H_3(z) = 4$$

$$G_0(z)H_0(W_3^i z) + G_1(z)H_1(W_3^i z) + G_2(z)H_2(W_3^i z) + G_3(z)H_3(W_3^i z) = 0 \quad \text{FOR } i=1,2,3.$$

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Patrick Naylor

(b)

$$G_2(z) = a + b z^{-1} + c z^{-2} + d z^{-3}$$

CONDITIONS:

$$\left.\begin{array}{l} g_2[n] \perp g_0[n] \\ g_2[n] \perp g_1[n] \end{array}\right. \Rightarrow \begin{cases} a+b = -(c+d) \\ a+b = c+d \end{cases}$$

THUS $a = -b$ AND $c = -d$

WE CHOOSE $a = c$ AND $b = d = -a$

CONDITION $\|g_2[n]\|^2 = 1$ LEADS TO $a = \frac{1}{2}$

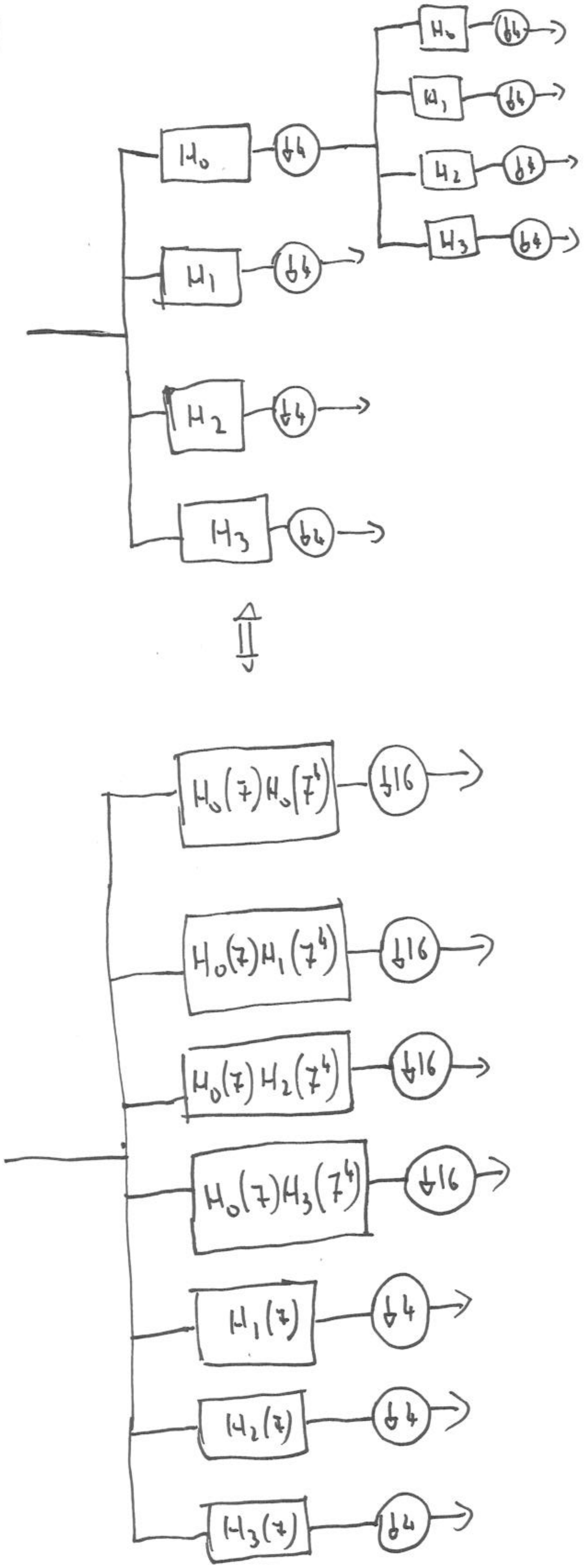
THUS

$$G_2(z) = \frac{1}{2}\left(1 - z^{-1} + z^{-2} - z^{-3}\right)$$

THE ~~THREE~~ FOUR CONDITIONS

$$\left.\begin{array}{l} g_3[n] \perp g_0[n] \\ g_3[n] \perp g_1[n] \\ g_3[n] \perp g_2[n] \\ \|g_3[n]\|^2 = 1 \end{array}\right. \Rightarrow G_3(z) = \frac{1}{2}\left(-1 + z^{-1} + z^{-2} - z^{-3}\right)$$

(c)



H0
↓4
H0
↓4
H1
↓4
H2
↓4
H3
↓4
H1
↓4
H2
↓4
H3
↓4
H0(z)H0(z^4)
↓16
H0(z)H1(z^4)
↓16
H0(z)H2(z^4)
↓16
H0(z)H3(z^4)
↓16
H1(z)
↓4
H2(z)
↓4
H3(z)
↓4

QUESTION 2

(a)

$$P(z) = \frac{1}{2\sqrt{2}} (1+z)^2 (1+z^{-1})^2 B(z)$$

IF $B(z) = a$ $\quad P(z) + P(-z) \neq 2$

SO LET'S TRY $B(z) = (a z^{-1} + b + a z)$

$$P(z) = \frac{1}{2\sqrt{2}} (1 + 2z + z^2)(1 + 2z^{-1} + z^{-2})(a z^{-1} + b + a z) =$$

$$= \left(a z^{-3} + (b+4a) z^{-2} + (7a + 4b) z^{-1} + (8a + 6b) + (7a + 4b) z + (b + 4a) z^{-2} + a z^3\right) / 2\sqrt{2}$$

THE HALF-BAND CONDITION $P(z) + P(-z) = 2$ IMPLIES THAT:

$$\left.\begin{array}{l} (b + 4a) = 0 \\ \frac{1}{2\sqrt{2}} (8a + 6b) = 1 \end{array}\right\} \Rightarrow \begin{array}{l} a = -\frac{2\sqrt{2}}{16} \\ b = \frac{2\sqrt{2}}{4} \end{array}$$

$$P(z) = \frac{1}{16} (1+z)^2 (1+z^{-1})^2 (-z^{-1} + 4 - z)$$

AND

$$H_0(z) = \frac{1}{4\sqrt{2}} (1+z)(1+z^{-1})(-z + 4 - z^{-1})$$

#

(b) $H_1(z) = z\, G_0(-z) = \frac{1}{2\sqrt{2}} z(1-z)(1-z^{-1})$

$$G_1(z) = z^{-1} H_0(-z) = \frac{z^{-1}}{4\sqrt{2}} (1-z)(1-z^{-1})(z+4+z^{-1})$$

(c) ROOTS OF z^2+4z+1 ARE

$$z_0 = -2-\sqrt{3}$$
$$z_1 = -2+\sqrt{3}$$

NOTICE THAT $z_0 = 1/z_1$.

THUS

$$P(z) = \frac{1}{16a}(1+z^2)(1+z^{-2})(1-a z^{-1})(1-a z)$$

WHERE $a = (-2+\sqrt{3})$

THUS

$$G_0(z) = \frac{1}{4\sqrt{a}}(1+z^{-2})(1-a z^{-1})$$

$$H_0(z) = G_0(z^{-1})$$
$$G_1(z) = -z^{-1} G_0(-z^{-1})$$
$$H_1(z) = G_1(z^{-1})$$

QUESTION 3

(a)

PARTITION OF UNITY

$$\sum_{m=-\infty}^{\infty} \varphi(t-m) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k)\, e^{j2\pi k t} = 1$$

IMPLIES THAT

$$\begin{cases} \hat{\varphi}(2\pi k) = 1 & \text{FOR } k=0 \\ \hat{\varphi}(2\pi k) = 0 & \text{FOR } k \neq 0 \text{ AND } k \in \mathbb{Z} \end{cases}$$

NOW, $\varphi_0(t)$ AND $\varphi_1(t)$ SATISFY PARTITION OF UNITY, MOREOVER $\varphi_2(t) = \varphi_0(t) * \varphi_1(t)$

$$\Updownarrow$$

$$\hat{\varphi}_2(w) = \hat{\varphi}_0(w)\, \hat{\varphi}_1(w)$$

THUS $\hat{\varphi}_2(2\pi k) = 1$ FOR $k=0$ AND $\hat{\varphi}_2(2\pi k) = 0$ $k \neq 0$

AND

$$\sum_{m} \varphi_2(t-m) = \sum_{k} \hat{\varphi}_2(2\pi k)\, e^{j2\pi k t} = 1$$

\#

(b) $\varphi_0(t)$ AND $\varphi_1(t)$ SATISFY THE TWO-SCALE RELATION THUS IN FOURIER DOMAIN WE HAVE THAT

$$\hat{\varphi}_0(w) = \frac{1}{\sqrt{2}}\, G_0\left(e^{\frac{jw}{2}}\right) \hat{\varphi}_0\left(\frac{w}{2}\right)$$

$$\hat{\varphi}_1(w) = \frac{1}{\sqrt{2}}\, G_1\left(e^{\frac{jw}{2}}\right) \hat{\varphi}_1\left(\frac{w}{2}\right)$$

AND

$$\hat{\varphi}_2(w) = \hat{\varphi}_0(w)\hat{\varphi}_1(w) = \frac{1}{\sqrt{2}}\left[\cdot\frac{1}{\sqrt{2}} G_0\left(e^{j\frac{w}{2}}\right) G_1\left(e^{j\frac{w}{2}}\right)\right]\hat{\varphi}_0\left(\frac{w}{2}\right)\hat{\varphi}_1\left(\frac{w}{2}\right) =$$

$$= \frac{1}{\sqrt{2}} G_2\left(e^{j\frac{w}{2}}\right)\hat{\varphi}_2\left(\frac{w}{2}\right) \quad \text{WITH} \quad g_2[n] = \frac{1}{\sqrt{2}} g_0[n] * g_1[n]$$

#

(c) IN THIS CASE

$$g_0[n] = \frac{1}{\sqrt{2}}\left(\delta[n] + \delta[n-1]\right) \Leftrightarrow G_0(z) = \frac{(1+z^{-1})}{\sqrt{2}}$$

$$g_1[n] = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]\right) \Leftrightarrow G_1(z) = \frac{(1+z^{-1})^2}{2\sqrt{2}}$$

THUS

$$g_2[n] = \frac{1}{\sqrt{2}} g_0[n] * g_1[n] \Leftrightarrow G_2(z)\ \frac{1}{\sqrt{2}}\ \frac{(1+z^{-1})^3}{4}$$

#

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QUESTION 4

(a)

$$\hat{X}(z) = \frac{1}{2}\left[G(z^{1/2})H(z^{1/2}) + G(-z^{1/2})H(-z^{1/2}) \right] X(z)$$

PR CONDITION:

$$G(z)H(z) + G(-z)H(-z) = 2 \quad \text{OR}$$

$$P(z) + P(-z) = 2$$

(b) $G(z) = a$ DOES NOT WORK

TRY $G(z) = a z^{-1} + b + a z$

$$P(z) = (z^{-2} + z^{-1} + 1 + z + z^{2})(a z^{-1} + b + a z) =$$

$$= a z^{-3} + (a+b) z^{-2} + (2a+b) z^{-1} + (2a+b) + (2a+b) z + (a+b) z^{2} + a z^{3}.$$

$$P(z) + P(-z) = 2 \Rightarrow \begin{cases} a+b = 0 \\ 2a+b = 1 \end{cases} \Rightarrow \begin{array}{l} a = 1 \\ b = -1 \end{array}$$

$$G(z) = (z^{-1} - 1 + z)$$

(c) $\hat{X}(z) = 0 \iff P(z) + P(-z) = 0$

THIS IS ACHIEVED WHEN $G(z) = (1 - z^{-1})$

IN THIS CASE WE HAVE

$$P(z) = (1 + z + z^2 + z^3)(1 - z^{-1}) = z^3 - z^{-1}$$

THUS

$$P(z) + P(-z) = 0$$

(d)

(i)

$$P_i = \prod_{k=0}^{i} a^{b^k} = a^{\sum_{k=0}^{i} b^k}$$

USING $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$ $\quad |\rho|<1$

WE OBTAIN

$$P = \lim_{i\to\infty} P_i = \lim_{i\to\infty} a^{\sum_{k=0}^{i} b^k} = a^{\sum_{k=0}^{\infty} b^k} = a^{\frac{1}{1-b}} \quad (1)$$

#

(ii)

$$\prod_{k=1}^{i} m_0\left(\frac{\omega}{2^k}\right) = \prod_{k=1}^{i} e^{-\frac{j\omega}{2^{k+1}}}\left(\frac{e^{\frac{j\omega}{2^{k+1}}} + e^{-\frac{j\omega}{2^{k+1}}}}{2}\right) =$$

$$= \prod_{k=1}^{i} e^{-\frac{j\omega}{2^{k+1}}} \prod_{k=1}^{i} \cos\left(\frac{\omega}{2^{k+1}}\right)$$

USING THE RESULT OF PART (i),

$$(a) \lim_{i\to\infty} \prod_{k=1}^{i} e^{-j\frac{\omega}{2^{k+1}}} = e^{-\frac{j\omega}{2}}$$

$$\prod_{k=1}^{i} \cos\frac{\omega}{2^{k+1}} = \prod_{k=1}^{i} \frac{\sin\left(\frac{\omega}{2^k}\right)}{2\sin\left(\frac{\omega}{2^{k+1}}\right)} = \frac{1}{2^i}\,\frac{\sin\frac{\omega}{2}}{\sin\left(\frac{\omega}{2^{i+1}}\right)}$$

$$(b) \lim_{i\to\infty} \frac{1}{2^i}\,\frac{\sin\frac{\omega}{2}}{\sin\left(\frac{\omega}{2^{i+1}}\right)} = \frac{\sin \omega/2}{\omega/2}$$

BY COMBINING (a) WITH (b) WE OBTAIN THE RESULT #