E4.45
C5.21
SO22
ISE4.47

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE/ISE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Wednesday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) : P.L. Dragotti
Second Marker(s) : P.A. Naylor

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

The Questions

1. Consider the three-channel filter bank shown in Figure 1

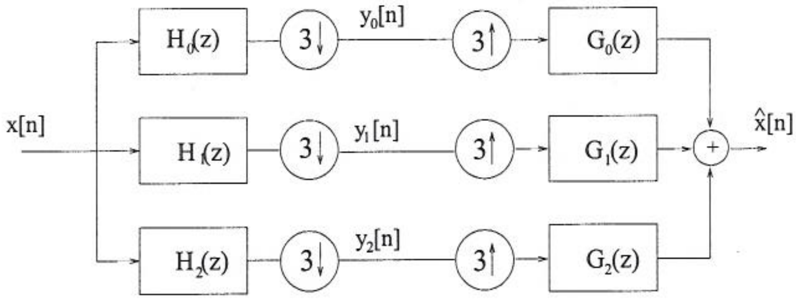


Figure 1: Three-channel filter bank.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the three perfect reconstruction conditions the filters have to satisfy.

[7]

(b) Assume that $G_0(z)$, $G_1(z)$ and $G_2(z)$ are $\frac{1}{3}$-band ideal filters as shown in Figure 2, and assume that $H_i(z) = G_i(z^{-1})$, for $i = 0, 1, 2$. Sketch and dimension the Fourier transform of $y_0[n], y_1[n], y_2[n]$ and $\hat{x}[n]$ assuming that $x[n]$ has the spectrum shown in Figure 3.

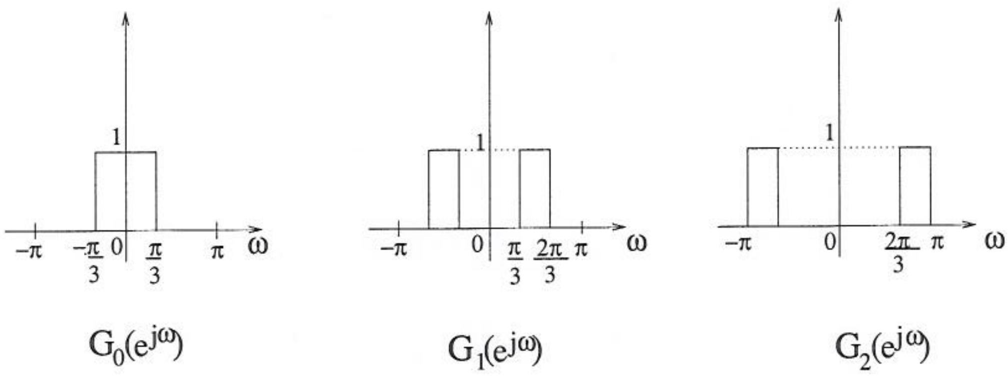


Figure 2: Fourier transforms of the synthesis filters of Figure 1.

[7]

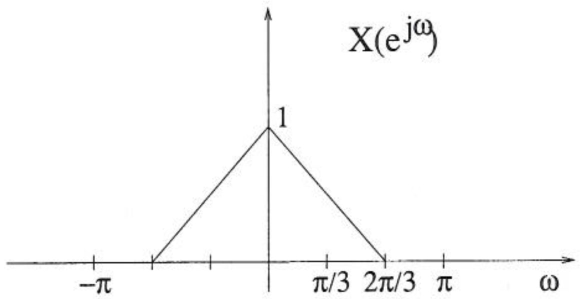


Figure 3: Spectrum of $x[n]$.

(c) Now, the filter bank is iterated on the H_0 branch to form a 2-level decomposition.

i. Draw either the synthesis or the analysis filter bank of the equivalent 5-channel filter bank clearly specifying the transfer functions and downsampling factors.

[3]

ii. If the filters are $\frac{1}{3}$-band and ideal as shown in Figure 2, draw the Fourier transform of the equivalent filters of each branch before downsampling.

[3]

2. Consider the two-channel filter bank of Figure 4.

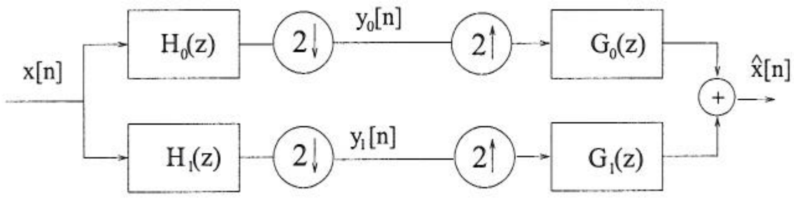


Figure 4: Two-channel filter bank.

(a) Assume that $G_0(z) = (1+z^{-1})(a+bz^{-1}+az^{-2})$ with $a \neq 0$ and $b \neq 0$. Find the values of a and b such that the condition $\langle g_0[n], g_0[n-2k] \rangle = \delta[k]$ is satisfied, where $\langle g_0[n], g_0[n-2k] \rangle$ denotes the inner product between $g_0[n]$ and $g_0[n-2k]$.

[5]

(b) Assume $G_0(z)$ is the filter you obtain in part (a). Design the filters $H_0(z), H_1(z), G_1(z)$ in order to have a perfect reconstruction orthogonal filter bank.

[5]

(c) Consider the two-channel filter bank of Figure 4 without downsamplers and upsamplers. Such a filter bank is called a *Nonsubsampled* filter bank. Choose $\{H_0(z), H_1(z), G_0(z), G_1(z)\}$ as in an orthogonal two-channel filter bank. What is $\hat{x}[n]$ as a function of $x[n]$ and the filters?

[5]

(d) Assume $H_0(z)$ and $G_0(z)$ are given, show how to obtain $H_1(z)$ and $G_1(z)$ such that $\hat{x}[n] = x[n]$ in the nonsubsampled filter bank. Assume that $H_0(z) = 1$ and $G_0(z) = z^{-1} + 4 + z$, calculate $\{H_1(z), G_1(z)\}$. Is the solution $\{H_1(z), G_1(z)\}$ unique? If not, provide at least two more alternative solutions.

[5]

3. Consider the two-channel filter bank of Figure 4 shown in Question 2.

(a) Take

$$P(z) = \frac{1}{16}(1+z)^2(1+z^{-1})^2(-z+4-z^{-1}),$$

where $P(z) = H_0(z)G_0(z)$. Compute a linear phase factorization of $P(z)$. That is, assume that $H_0(z) = \frac{1}{4\sqrt{2}}(1+z^{-1})^2(1+z)$. Given this choice of $H_0(z)$, define the other filters $H_1(z), G_0(z)$ and $G_1(z)$ in terms of their z-transforms.

[7]

(b) Now, consider the two limit functions

$$\hat{\varphi}(\omega) = \lim_{J\to\infty}\prod_{k=1}^{J} M_0\left(\frac{\omega}{2^k}\right),$$

$$\hat{\tilde{\varphi}}(\omega) = \lim_{J\to\infty}\prod_{k=1}^{J} \tilde{M}_0\left(\frac{\omega}{2^k}\right),$$

where $M_0(\omega) = \frac{G_0(e^{j\omega})}{\sqrt{2}}$, $\tilde{M}_0(\omega) = \frac{H_0(e^{j\omega})}{\sqrt{2}}$ and $\hat{\varphi}(\omega), \hat{\tilde{\varphi}}(\omega)$ are the Fourier transforms of $\varphi(t)$ and $\tilde{\varphi}(t)$ respectively. What can you say about convergence, continuity and differentiability of $\varphi(t)$ and $\tilde{\varphi}(t)$?

[7]

(c) Assume that the two limit functions $\varphi(t)$ and $\tilde{\varphi}(t)$ exist and that $\varphi(t)$ and $\tilde{\varphi}(t)$ are two valid scaling functions. Consider the two corresponding wavelets

$$\psi(t) = \sqrt{2}\sum_n h_1[n]\varphi(2t-n) \text{ and } \tilde{\psi}(t) = \sqrt{2}\sum_n g_1[n]\tilde{\varphi}(2t-n),$$

where $h_1[n]$ and $g_1[n]$ are the filters you found in (a).

i. State the number of vanishing moments of $\psi(t)$ and $\tilde{\psi}(t)$.

[3]

ii. Consider a function $f(t) \in L_2(\mathbb{R})$ and assume $f(t)$ is uniformly α-Lipschitz with $\alpha = 2.2$. You can write $f(t)$ either in terms of $\psi(t)$ or $\tilde{\psi}(t)$. That is:

$$f(t) = \sum_m \sum_n \langle f(t), \tilde{\psi}_{m,n}(t) \rangle \psi_{m,n}(t)$$

or

$$f(t) = \sum_m \sum_n \langle f(t), \psi_{m,n}(t) \rangle \tilde{\psi}_{m,n}(t),$$

with the usual assumption that $\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n)$. Which of these two representations leads to a faster decay of the wavelet coefficients across scales? Justify your answer numerically. In the light of these considerations, discuss whether or not it would be better to exchange the roles of the analysis and synthesis filters.

[3]

4. Consider the linear B-Spline given by

$$\varphi(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ \\ 0, & \text{otherwise.} \end{cases}$$

We know that $\varphi(t)$ is a valid scaling function. However, the linear spline is not orthogonal. It is our aim to orthogonalize it.

(a) Compute the deterministic autocorrelation function

$$a[n] = \langle \varphi(t), \varphi(t-n) \rangle.$$

Denote $\hat{\varphi}(\omega)$ to be the Fourier transform of $\varphi(t)$ and $A(e^{j\omega})$ to be the discrete-time Fourier transform of $a[n]$. Show that the new function $\phi(t)$ with Fourier transform

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}}$$

is an orthogonal basis of the subspace $V_0 = \text{ span } \{\phi(t-n)\}_{n\in\mathbb{Z}}$. (Hint: Show that the Riesz basis criterion $A \leq \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 \leq B$ is satisfied with $A = B = 1$).

[5]

(b) Using the Poisson sum formula:

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi kt},$$

show that $\phi(t)$ satisfies the partition of unity.

[5]

(c) Finally, find the z-domain expression of the filter $H_0(z)$ that leads to the two-scale equation:

$$\phi(t) = \sqrt{2}\sum_n h_0[n]\phi(2t-n).$$

(Hint: Use the fact that $\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}}\hat{\varphi}(\omega/2)$ and the fact that $\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}}$.)

[5]

(d) You now have a valid orthogonal scaling function, find the corresponding wavelet. That is, find the z-domain expression of the filter $H_1(z)$ such that: $\psi(t) = \sqrt{2}\sum_n h_1[n]\phi(2t-n)$.

[5]

QUESTION 1

E4.45
ISE 4.47
SO22
C5.21

(a)

$$\hat{X}(z) = \frac{1}{3}\Big[G_0(z)\Big(X(z)H_0(z) + X(W_3^1 z)H_0(W_3^1 z) + X(W_3^2 z)H_0(W_3^2 z)\Big) + G_1(z)\Big(X(z)H_1(z) + X(W_3^1 z)H_1(W_3^1 z) + X(W_3^2 z)H_1(W_3^2 z)\Big) + G_2(z)\Big(X(z)H_2(z) + X(W_3^1 z)H_2(W_3^1 z) + X(W_3^2 z)H_2(W_3^2 z)\Big)\Big].$$

THUS FOR PERFECT RECONSTRUCTION WE REQUIRE:

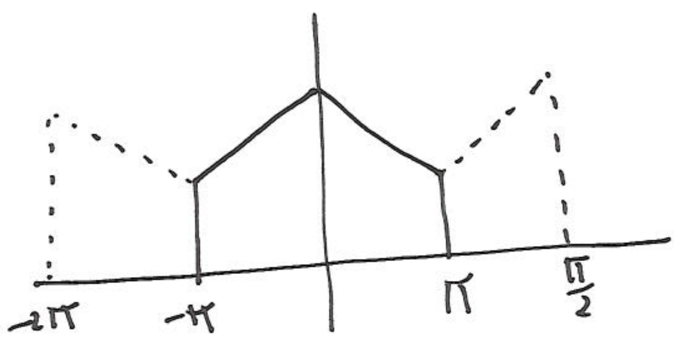
$$G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z) = 3$$

AND THE TWO FOLLOWING NO-ALIASING CONDITIONS:

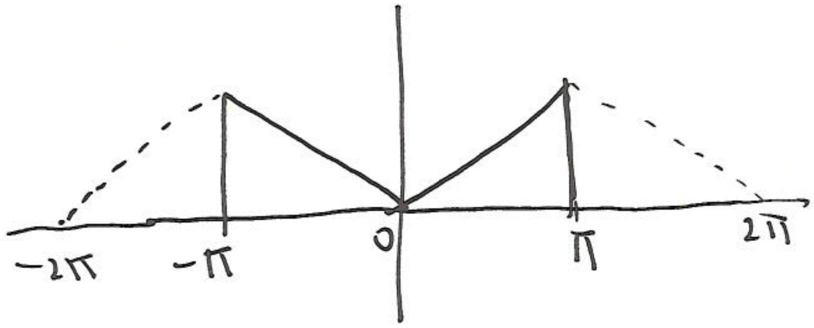
$$\begin{cases} G_0(z)H_0(W_3^1 z) + G_1(z)H_1(W_3^1 z) + G_2(z)H_2(W_3^1 z) = 0 \\ G_0(z)H_0(W_3^2 z) + G_1(z)H_1(W_3^2 z) + G_2(z)H_2(W_3^2 z) = 0 \end{cases}$$

WHERE $W_N^k = e^{-j\frac{2\pi k}{N}}$

(b) $Y_0(w)$



$Y_1(e^{jw})$

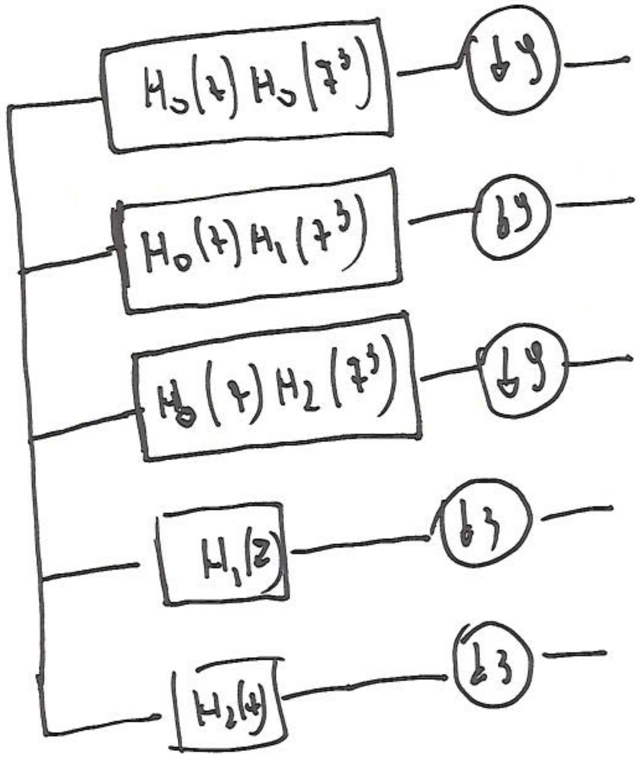


$Y_2(e^{jw}) = 0$

$\hat{X}(e^{jw}) = X(e^{jw})$

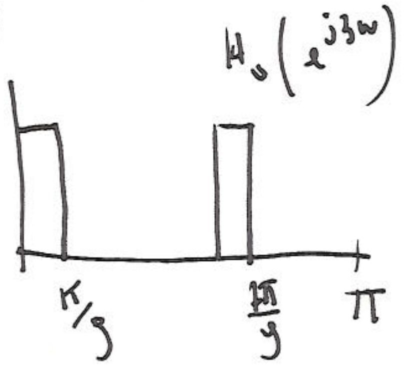
(c)

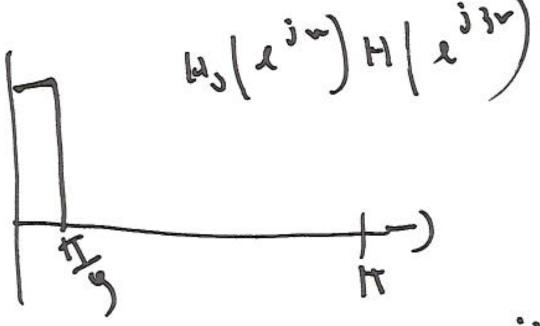
(i)

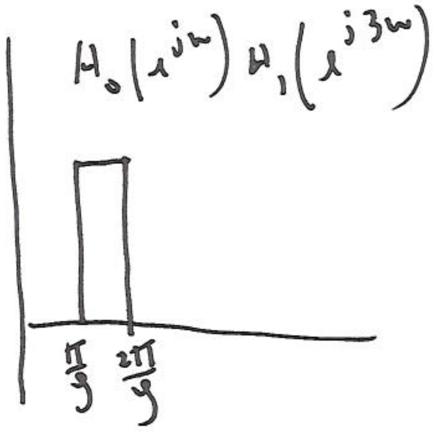


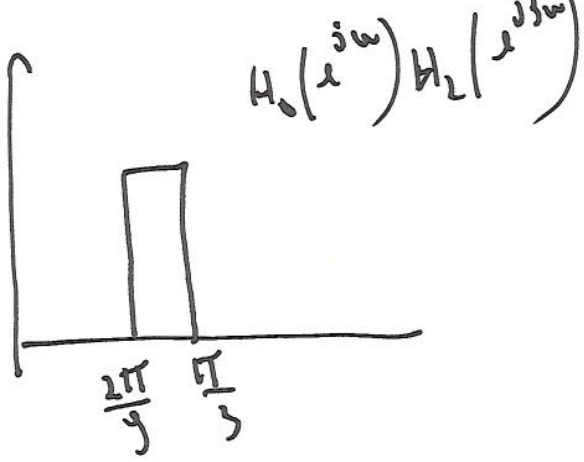
(ii)

(ii)


$H_0(e^{j3w})$
$\pi/9$
$3\pi/9$
π


$H_0(e^{jw})H(e^{j3w})$
$\pi/9$
π


$H_0(e^{jw})H_1(e^{j3w})$
$\frac{\pi}{9}$
$\frac{2\pi}{9}$


$H_0(e^{jw})H_2(e^{j3w})$
$\frac{2\pi}{9}$
$\frac{\pi}{3}$

QUESTION 2

(a) $G_0(z) = a + (a+b)z^{-1} + (a+b)z^{-2} + a z^{-3}$;

THE CONDITION

$$\langle g_0[n], g_0[n] \rangle = 1 \iff 2a^2 + 2(a+b)^2 = 1 \quad (1)$$

AND THE CONDITION

$$\langle g_0[n], g_0[n-2] \rangle = 0 \iff 2a(a+b) = 0$$

SINCE $a \neq 0$ AND $b \neq 0$ WE HAVE $a = -b$ AND FROM (1) WE GET

$$a = \frac{1}{\sqrt{2}}, \quad b = -\frac{1}{\sqrt{2}}$$

AND

$$G_0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-3}$$

(b)

$$G_1(z) = -z^{-1} G_0(-z^{-1}) = \frac{1}{\sqrt{2}} z^{2} - \frac{1}{\sqrt{2}} z^{-1}$$

$$H_0(z) = G_0(z^{-1}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{3}$$

$$H_1(z) = G_1(z^{-1}) = \frac{1}{\sqrt{2}} z^{-2} - \frac{1}{\sqrt{2}} z$$

(c)

$$\hat{X}(z) = \{H_0(z)G_0(z) + H_1(z)G_1(z)\}\, X(z)$$

SINCE $\{H_0, G_0, H_1, G_1\}$ ARE ORTHOGONAL, IT FOLLOWS THAT $\hat{X}(z) = 2X(z)$.

(d) $\hat{X}(z) = X(z) \Rightarrow H_0(z)G_0(z) + H_1(z)G_1(z) = 1$.

IF $H_0(z)$ AND $G_0(z)$ ARE GIVEN, WE HAVE THAT $H_1(z)G_1(z) = 1 - H_0(z)G_0(z)$.

IN THE EXAMPLE

$$H_1(z)G_1(z) = 1 - z^{-1} - 4 - z = -z^{-1} - 3 - z =$$

$$= z^{-1}(z - a_0)(z - a_1) \quad \text{WITH} \quad \begin{cases} a_1 = \dfrac{-3+\sqrt{5}}{2} \\ a_0 = \dfrac{-3-\sqrt{5}}{2} \end{cases}$$

THUS POSSIBLE SOLUTIONS ARE

$H_1(z) = 1$, $G_1(z) = z^{-1}(z - a_0)(z - a_1)$

$H_1(z) = (z - a_0)$, $G_1(z) = (z - a_1)z^{-1}$

$H_1(z) = (z^{-1} - a_1)z^{-1}$, $G_1(z) = (z - a_0)$

$H_1(z) = z^{-1}(z - a_0)(z - a_1)$, $G_1(z) = 1$

QUESTION 3

(a)

$$H_0(z) = \frac{1}{4\sqrt{2}}\left(1+z^{-1}\right)^2\left(1+z\right), \quad G_0(z) = \frac{1}{2\sqrt{2}}\left(1+z\right)\left(-z^{-1}+4-z\right)$$

$$H_1(z) = z\,G_0(-z), \quad G_1(z) = z^{-1}H_0(-z)$$

(b) BOTH FUNCTIONS SATISFY THE NECESSARY CONVERGE CONDITIONS.

USING DAUBECHIES CRITERION WE HAVE THAT

$$\tilde{H}_0(w) = \left(\frac{1+e^{-jw}}{2}\right)^3 \Rightarrow$$

$$N=3 \qquad B=1 \qquad B < 2^{N-1-p} = 2^{3-1-p}$$

THUS $p=1$ AND $\varphi(t) \in C^{(1)}$ THAT IS $\varphi(t)$ IS CONTINUOUS ~~AND~~ WITH ITS FIRST ORDER DERIVATIVE. IN FACT $\varphi(t)$ IS A QUADRATIC SPLINE.

$$H_0(w) = \left(\frac{1+e^{jw}}{2}\right)R(w) \quad \text{WHERE} \quad R(w) = \frac{\left(4-2\cos w\right)}{2}$$

$$B = \max_{w} R(w) = 3$$ THUS SUFFICIENT REGULARITY CONDITIONS ARE NOT SATISFIED. WE CANNOT GUARANTEE CONVERGENCE, ~~AND~~ AND REGULARITY

(c)

(i)

$$\hat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}} H_1\left(e^{j\frac{\omega}{2}}\right) \hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right)$$

$\hat{\tilde{\varphi}}(\omega) \neq 0$ FOR $\omega = 0$. THAT IS BECAUSE $\varphi(t)$ IS A VALID SCALING FUNCTION.

$H_1\left(e^{j\frac{\omega}{2}}\right)$ HAS THREE ZEROS AT $\omega = 0$.

THEREFORE $\tilde{\psi}(t)$ HAS THREE VANISHING MOMENTS.

THE OTHER WAVELET, $\psi(t)$, HAS ONE VANISHING MOMENT.

(ii)

$\langle f(t), \tilde{\psi}_{m,n}(t) \rangle$ DECAYS AS

$$2^{+m(\alpha+1/2)} = 2^{m(2.7)}$$

$\langle f(t), \psi_{m,n}(t) \rangle$ DECAYS AS

$$2^{m(1+1/2)} = 2^{1.5m}.$$

THUS THE DECOMPOSITION

$f(t) = \sum_m \sum_n \langle f, \tilde{\psi}_{m,n} \rangle \psi_{m,n}$ IS BETTER

AND THE ROLES OF ANALYSIS AND SINTHESIS FILTERS SHOULD BE SWAPPED.

QUESTION 4

(a)

$$a[n] = \begin{cases} 2/3 & \text{FOR } n = 0 \\ 1/6 & \text{FOR } n = \pm 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$A\left(e^{jw}\right) = \sum_{k=-\infty}^{\infty} \left| \hat{\varphi}(w + 2k\pi) \right|^2 \qquad (1)$$

THUS, IF $\quad \hat{\phi}(w) = \dfrac{\hat{\varphi}(w)}{\sqrt{A(e^{jw})}} \qquad (2)$

WE HAVE THAT

$$\sum_{k=-\infty}^{\infty} \left| \hat{\phi}(w + 2k\pi) \right|^2 \overset{(a)}{=} \sum_{k=-\infty}^{\infty} \left| \frac{\hat{\varphi}(w+2k\pi)}{\sqrt{A(e^{jw})}} \right|^2 =$$

$$= \frac{1}{A(e^{jw})} \sum_{k=-\infty}^{\infty} \frac{\left| \hat{\varphi}(w + 2k\pi) \right|^2}{\cancel{A(e^{jw})}} \overset{(b)}{=} 1$$

WHERE (a) FOLLOWS FROM EQ. (2) AND FROM THE FACT THAT $A(e^{jw})$ IS PERIODIC OF PERIOD 2π, AND (b) FOLLOWS FROM (1).

(b) FIRST NOTICE THAT $A\left(e^{j2\pi k}\right) = \cancel{A(e^{j0})}\, A(1) = 1$ AND THAT SINCE $\varphi(t)$ IS A VALID SCALING FUNCTION WE HAVE THAT:

$$\sum_n \varphi(t-n) = \sum_k \hat{\varphi}(2\pi k)\, e^{j2\pi k t} = 1 .$$

FOR THESE REASONS IT FOLLOWS THAT

$$\sum_n \phi(t-n) = \sum_k \frac{\hat{\varphi}(2\pi k)}{A(e^{j2\pi k})}\, e^{j2\pi k t} =$$

$$= \sum_k \hat{\varphi}(2\pi k)\, e^{j2\pi k t} = 1$$

(c) $$\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}}\, \hat{\varphi}(\omega/2) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}} = \frac{G_0(e^{j\omega/2})}{\sqrt{2\, A(e^{j\omega})}}\, \hat{\varphi}\left(\frac{\omega}{2}\right) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2}) \sqrt{\frac{A(e^{j\frac{\omega}{2}})}{A(e^{j\omega})}} \cdot \hat{\phi}\left(\frac{\omega}{2}\right) .$$

THUS

$$H_0(e^{j\omega/2}) = G_0(e^{j\frac{\omega}{2}}) \sqrt{\frac{A(e^{j\frac{\omega}{2}})}{A(e^{j\omega})}} \quad \text{AND}$$

$$H_0(z) = G_0(z) \cdot \sqrt{\frac{A(z)}{A(z^2)}}$$

NOW

$$G_0(z) = \frac{1}{\sqrt{2}}\left(\frac{1}{2}z^{-1} + 1 + \frac{1}{2}z\right) = \frac{1}{2\sqrt{2}}(1+z)(1+z^{-1})$$

AND

$$A(z) = \frac{1}{3}\left(\frac{1}{2}z^{-1} + 2 + \frac{1}{2}z\right)$$

THUS

$$H_0(z) = \frac{1}{2\sqrt{2}}(1+z)(1+z^{-1})\sqrt{\frac{(z^{-1}+4+z)}{(z^{-2}+4+z^2)}}$$

(d)

$$H_1(z) = -z^{-1}H_0(-z^{-1})$$