E4.45
SO22
ISE4.47

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

WAVELETS AND APPLICATIONS

Wednesday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	R. Nabar

Special Information for the Invigilators: NONE

Information for Candidates: NONE

The Questions

1. Multi-rate signal processing

   (a) Consider the following system:

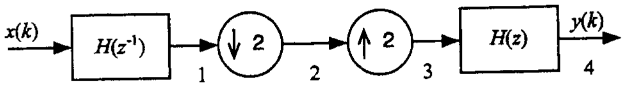


Figure 1: Multi-rate system.

Give the z-transform and Fourier transform of the signal at locations 1-4. Make the corresponding graphs of the Fourier transform assuming that $H(z)$ is an ideal half-band lowpass filter and that $X(z)$ has the following spectrum:

[6]

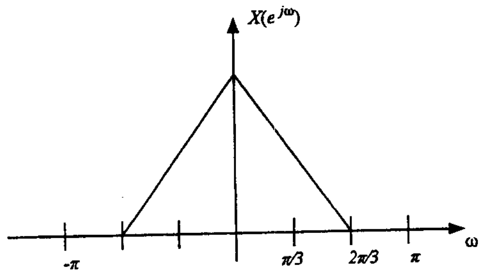


Figure 2: Spectrum of $x[k]$.

   (b) A transmultiplexer is the dual of a subband coder. Two signals are multiplexed and sent over a high bandwidth channel. A perfect reconstruction (PR) multiplexer cancels crosstalk and reconstruct the signals exactly.

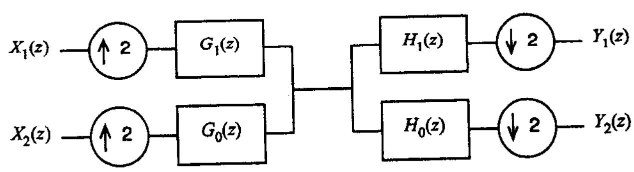


Figure 3: The Transmultiplexer.

i. Give the input/output relations in the z-transform domain. What are the conditions on the filters that guarantee that the trans-multiplexer is PR?

[7]

ii. Suppose that you have a power complementary filter $G_0(z)$ (i.e., $g_0[n]$ is such that $\langle g_0[n], g_0[n-2k] \rangle = \delta_k$). How can you use it to get a PR trasmultiplexer? Specify all four filters in terms of this prototype.

[7]

2. Spectral factorization method for two-channel filter banks. Consider the

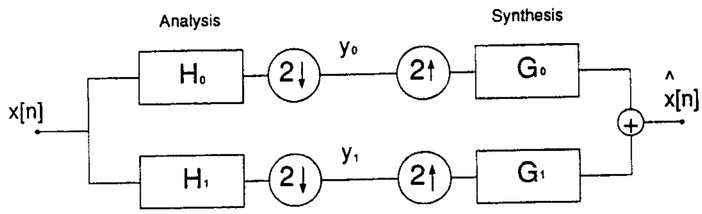


Figure 4: Two-channel filter bank.

factorization of $P(z)$ in order to obtain orthogonal or biorthogonal filter banks.

(a) Take
$$P(z) = (\frac{z^3}{2} + 1 + \frac{z^{-3}}{2}).$$
Compute a linear phase factorization of $P(z)$. In particular, assume that $H_0(z) = (z - 1 + z^{-1})$. Given this choice of $H_0(z)$, give the other filters of this biorthogonal filter bank.

[10]

(b) Now build an orthogonal filter bank based on this $P(z)$. (Hint: Remember that, if z_k is a root of $P(z)$ so is $1/z_k$, z_k^* and $1/z_k^*$).

[10]

3. Consider the linear B-Spline given by

$$\varphi(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $\varphi(x)$ is a valid scaling function. That is, show that

i. it satisfies the two scale equation $\varphi(x/2) = \sqrt(2) \sum_{n\in\mathbb{Z}} g[n]\varphi(x - n)$,

[5]

ii. it satisfies the partition of unity $\sum_{n\in\mathbb{Z}} \varphi(x-n) = 1$,

[5]

iii. it satisfies the Riesz basis criterion $0 < A \leq \sum_{k\in\mathbb{Z}} |\Phi(\omega+2\pi k)|^2 \leq B < \infty$.

[5]

(b) Now consider the derivative of $\varphi(x)$. Show that the derivative of $\varphi(x)$ is not a valid scaling function. (Hint: it is enough to show that at least one of the above criteria is not satisfied).

[5]

4. Consider the wavelet series expansion of continuous-time signals with the Haar wavelet $\psi(t)$.

   (a) Give the expansion coefficients

   $$d_{m,n} = \langle \psi_{m,n}, f \rangle$$

   for $f(t) = 1$, $t \in [0, 1]$, and 0 otherwise (that is, $f(t)$ is the Haar scaling function).

   [5]

   (b) Verify that $\sum_m \sum_n |\langle \psi_{m,n}, f \rangle|^2 = \|f(t)\|^2$.

   [5]

   (c) Now consider $g(t) = f(t - 2^{-i})$ where i is a positive integer. Give the range of scale over which expansion coefficients $d_{m,n} = \langle \psi_{m,n}, g \rangle$ are different from zero.

   [5]

   (d) Assume now that $f(t) = 1$ $t \in [0, 2]$ and 0 otherwise. Can $f(t)$ be considered a valid scaling function?

   [5]

WAVELETS + APPLICATIONS   E4.45 / SO22 / ISE 4.47

SOLUTIONS - 2005

1) MULTI-RATE SIGNAL PROCESSING

(a)

① $X(z)H(z^{-1})$

② $\frac{1}{2}\left(X(z^{1/2})H(z^{-1/2}) + X(-z^{1/2})H(-z^{-1/2})\right)$

③ $\frac{1}{2}\left(X(z)H(z^{-1}) + X(-z)H(-z)\right)$

④ $\frac{1}{2}H(z)\left[X(z)H(z^{-1}) + X(-z)H(-z^{-1})\right]$

IN FOURIER DOMAIN

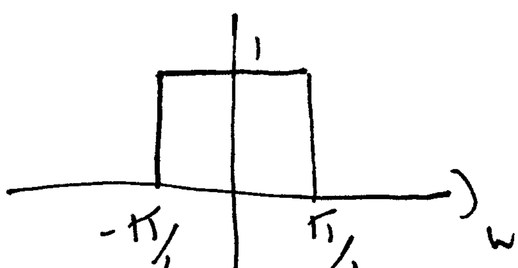
① $X(e^{j\omega})H(e^{-j\omega})$

② $\frac{1}{2}\left(X(e^{j\omega/2})H(e^{-j\omega/2}) + X\left(e^{j\left(\frac{\omega}{2}+\pi\right)}\right)H\left(e^{-j\left(\frac{\omega}{2}+\pi\right)}\right)\right)$

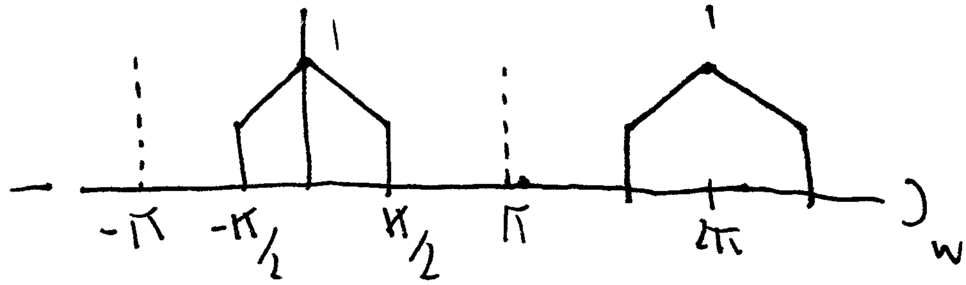
③ $\frac{1}{2}\left[X(e^{j\omega})H(e^{-j\omega}) + X(e^{j\omega+\pi})H(e^{-j\omega+\pi})\right]$

④ $\frac{1}{2}H(e^{j\omega})\left[X(e^{j\omega})H(e^{-j\omega}) + X(e^{j(\omega+\pi)})H(e^{-j\omega+\pi})\right]$

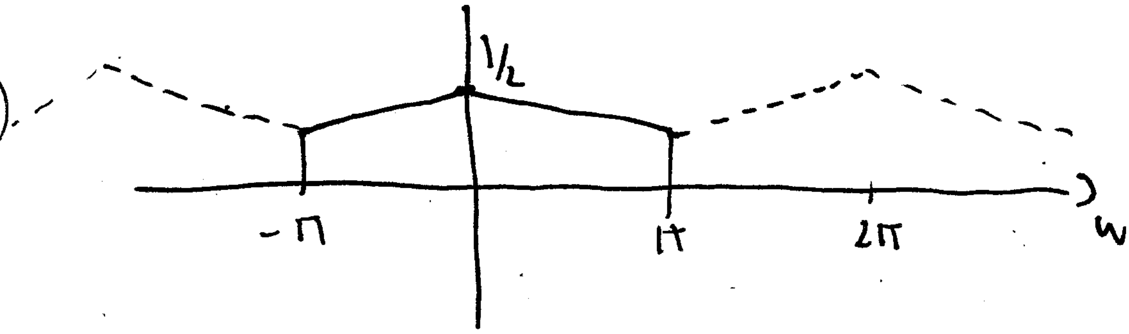
$H(e^{j\omega}) = H(e^{-j\omega})$



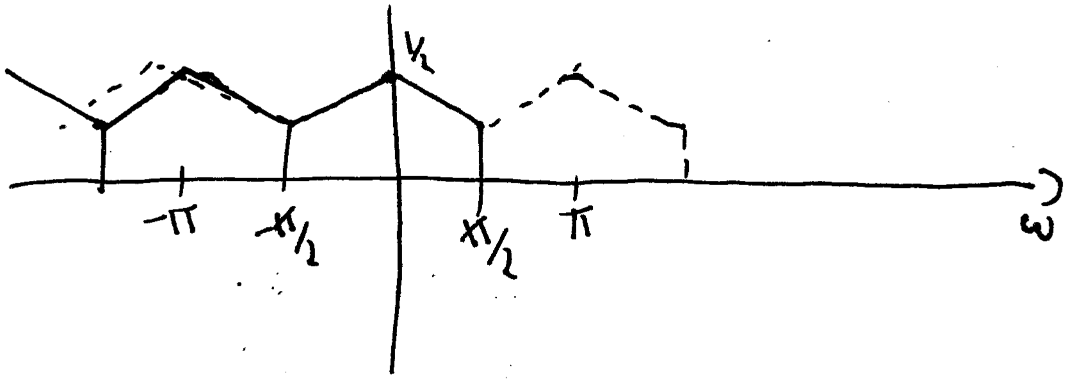
(1)



(2)



(3)



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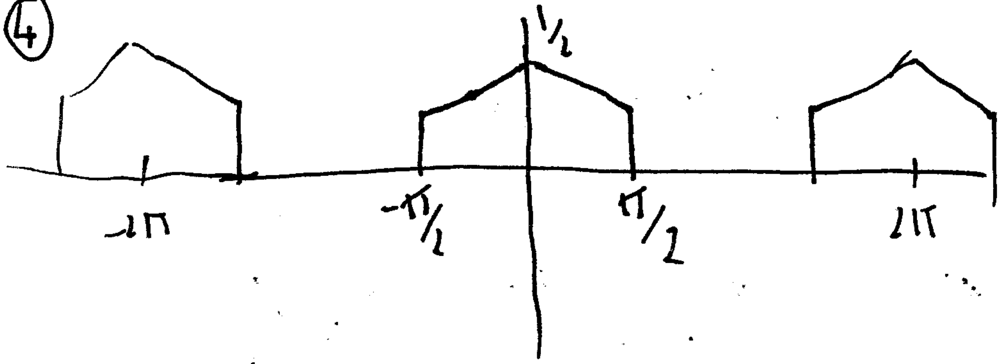
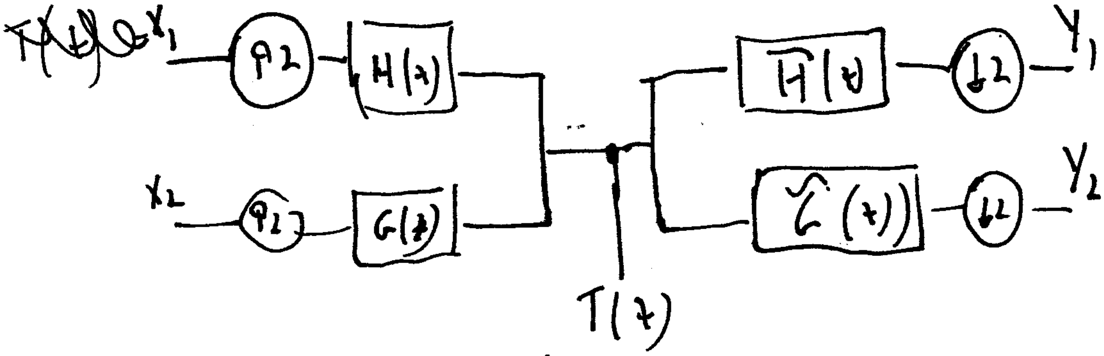


PART (b)



$$T(z) = X_1(z^2)\,H(z) + X_2(z^2)\,G(z)$$

$$Y_1(z) = \frac{1}{2}\left[T(z^{1/2})\tilde{H}(z^{1/2}) + T(-z^{1/2})\tilde{H}(-z^{1/2})\right]$$

$$Y_2(z) = \frac{1}{2}\left[T(z^{1/2})\tilde{G}(z^{1/2}) + T(-z^{1/2})\tilde{G}(-z^{1/2})\right]$$

$$Y_1(z^2) = \frac{1}{2}\Big[X_1(z^2)H(z)\tilde{H}(z) + X_2(z^2)G(z)\tilde{H}(z) + X_1(z^2)H(-z)\tilde{H}(-z) + X_2(z^2)G(-z)\tilde{H}(-z)\Big]$$

$$Y_2(z^2) = \frac{1}{2}\Big[X_1(z^2)H(z)\tilde{G}(z) + X_2(z^2)G(z)\tilde{G}(z) + X_1(z^2)H(-z)\tilde{G}(-z) + X_2(z^2)G(-z)\tilde{G}(-z)\Big]$$

PR : $Y_1(z) = X_1(z)$ & $Y_2(z) = X_2(z)$

$$\Rightarrow \begin{cases} H(z)\tilde{H}(z) + H(-z)\tilde{H}(-z) = 2 \\ G(z)\tilde{G}(z) + G(-z)\tilde{G}(-z) = 2 \end{cases}$$

NO CROSS-TALK

$$\begin{cases} G(z)\tilde{H}(z) + G(-z)\tilde{H}(-z) = 0 \\ \tilde{G}(z)H(z) + \tilde{G}(-z)H(-z) = 0 \end{cases}$$

ii. AS TRASMULTIPLEXER IS STRUCTURAL EQUIVALENT TO 2-CHANNEL PR FILTER BANK, WE HAVE THAT

$$\begin{cases} G(z)\,G(z^{-1}) + G(-z)\,G(-z^{-1}) = 2 \\ \tilde{G}(z) = G(z^{-1}) \\ \cancel{\tilde{H}(z) = -z^{-1}\,\tilde{G}(-z^{-1})} \\ H(z) = -z^{-1}\,G(-z^{-1}) \\ \tilde{H}(z) = H(z^{-1}) \end{cases}$$

2\. (a) $P(z) = H_0(z)\,G_0(z)$

IF $H_0(z) = (z - 1 + z^{-1})$

THEN $G_0(z) = \left(\frac{1}{2}z^{-2} + \frac{1}{2}z^{-1} + \frac{1}{2}z + \frac{1}{2}z^{2}\right)$

THE OTHER TWO FILTERS ARE

$$G_1(z) = z^{-1}H_0(-z) \qquad H_1(z) = z\,G_0(-z)$$

(b) $P(z) = \frac{1}{2}(z - 1 + z^{-1})(z - 1 + z^{-1})(1+z)(1+z^{-1})$

THEREFORE

$$G_0(t) = \frac{1}{\sqrt{2}}\left(1+t^{-1}\right)\left(t-1+t^{-1}\right)$$

$H_0(t)$ MUST BE EQUAL TO $G_0(t^{-1})$

IN FACT $\quad H_0(t) = \frac{1}{\sqrt{2}}(1+t)\left(t-1+t^{-1}\right)$

$$G_1(t) = -t^{-1}G_0(t^{-1}) \quad \text{AND} \quad H_1(t) = G_1(t^{-1})$$

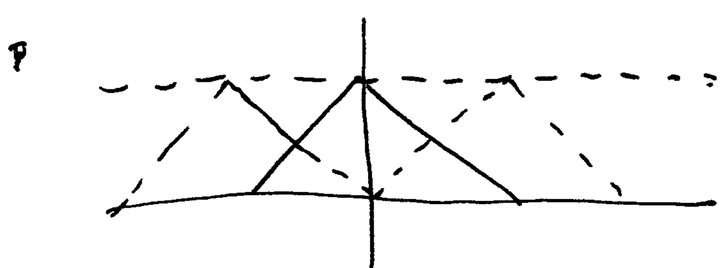
3)

(a)

i TWO SCALE RELATION IS SATISFIED FOR

$$\begin{cases} g_0[1] = g_0[-1] = \frac{1}{2\sqrt{2}} \\ g_0[0] = \frac{1}{\sqrt{2}} \\ g_0[n] = 0 \quad \text{OTHERWISE} \end{cases}$$

ii)



$\varphi(x)$ CLEARLY SATISFIES PARTITION OF UNITY iii)

$$x[n] = \langle \varphi(x), \varphi(x-n) \rangle = \begin{cases} 1 & \text{FOR } n=0 \\ \frac{1}{6} & \text{FOR } n=\pm 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\sum_k \left| \Phi(\omega + 2k\pi) \right|^2 = \sum_n x[n] e^{-j\omega n} = 1 + \frac{1}{3}\cos\omega$$

THUS

$$A = 1 - \frac{1}{3} = \frac{2}{3} > 0$$

$$B = 1 + \frac{1}{3} = \frac{4}{3} < +\infty$$

(b) THE DERIVATIVE OF $\varphi(x)$ DOES NOT SATISFY PARTITION OF UNITY THUS IT IS <u>NOT</u> A VALID SCALING FUNCTION

4. (a)

$$\psi_{m,n}(t) = \frac{1}{\sqrt{2^m}} \psi\left(2^{-m} t - n\right)$$

$$\alpha_{m,n} = \begin{cases} \cancel{0 \quad \text{FOR } m \le 0} \\ \frac{1}{\sqrt{2^m}} & \text{FOR } m > 0 \ \& \ n=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

(b) $\|f\|^2 = 1$

$$\sum_n \sum_m |\langle \psi_{m,n}, f \rangle|^2 = \sum_{m \geq 1} |d_{m,n}|^2 =$$

$$= \sum_{m=1}^{\infty} \frac{1}{2^m} = \frac{1}{1-1/2} - 1 = 1$$

(c) FROM $m=-i$ TO $m=+\infty$

(d) $f(t) = 1 \quad t \in [0,2]$ IS NOT A VALID SCALING FUNCTION SINCE IT DOES NOT SATISFY PARTITION OF UNITY