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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

MSc and EEE PART IV: M.Eng. and ACGI

OPTIMIZATION

Thursday, 15 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) : A. Astolfi
Second Marker(s) : J.M.C. Clark

Special instructions for invigilators: None

Information for candidates:

- All functions are sufficiently smooth.
- ∇f denotes the gradient of the function f. Note that ∇f is a column vector.
- $\nabla^2 f$ denotes the Hessian matrix of the function f. Note that this is a symmetric square matrix.
- Let $f : I\!R^n \to I\!R$. A level set of f is any non-empty set described by

$$\mathcal{L}(\alpha) = \{x \in I\!R^n \ : \ f(x) \leq \alpha\},$$

with $\alpha \in I\!R$.

1. Consider the problem of minimizing the function

$$f(x) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 2)$$

(a) Compute the stationary points of the function. [4]

(b) Using second order sufficient conditions *classify* the stationary points determined in part (a), *i.e.* say which is a local minimum, a local maximum and a saddle point. [6]

(c) Consider the minimization of the function f using the gradient algorithm. Express analytically the form of the generic iteration, *i.e.*

$$p_{k+1} = p_k - \alpha \nabla f \qquad (\star)$$

(where $p_i = [x_1^i, x_2^i]^T$). [2]

(d) Equation $(\star)$ defines a nonlinear discrete time system with equilibria coinciding with the stationary points of the function f.

Consider the linear approximation of system $(\star)$ around the equilibrium corresponding to the local minimum of the function f. Show that there exist a value of $\alpha > 0$ such that the eigenvalues of such a linearized system are both in modulus smaller than 1. (Hint: try a small positive α.)

Interpret the obtained result in terms of convergence properties of sequences generated by the gradient algorithm and initialized close to a local minimum. [8]

2. (a) Describe how Newton's method can be used to compute the solutions of a system of nonlinear equations. [6]

(b) Consider the equation

$$x^2 + 2bx + c = 0$$

and assume $b^2 - c > 0$. Using the results in part (a), write the Newton's iteration for the computation of solutions of the considered equation. Show that the iteration is well defined, *i.e.* if x_0 is such that $x_0 + b > 0$ then $x_k + b > 0$ for all k, whereas if x_0 is such that $x_0 + b < 0$ then $x_k + b < 0$ for all k. [8]

(c) Consider the problem of computing an approximate numerical value of $\sqrt{3}$. This is equivalent to computing an approximate solution of the equation $x^2 - 3 = 0$. Write the Newton's iteration associated to this problem. Let $\{x_k\}$ be the sequence generated by the algorithm initialized at $x_0 = 1$. Evaluate numerically the elements x_1, x_2, x_3 and x_4 of this sequence. Show, computing the relative approximation error

$$\frac{x_4 - \sqrt{3}}{\sqrt{3}},$$

that x_4 is a *very accurate* approximation of $\sqrt{3}$. [6]

3. Consider the minimization problem

$$\begin{cases} \min_{x} x'x + 2d'x \\ x'x - a^2 = 0 \end{cases}$$

with $x \in I\!R^n$, $a > 0$ and the vector $d \neq 0$.

(a) Write the first order necessary conditions of optimality for such a constrained optimization problem. [4]

(b) Using the conditions in part (a) compute candidate optimal solutions. Compute the corresponding optimal multiplier. Finally show, using second order sufficient conditions, which of the obtained candidate solutions is a local minimum. [8]

(c) Suppose $x \in I\!R^2$, *i.e.* $x = (x_1, x_2)$. Set $x_1 = a\cos\theta$ and $x_2 = a\sin\theta$, with $\theta \in (-\pi, \pi]$. Show that with the above selection the constraint $x'x - a^2 = 0$ is *automatically* satisfied. Then, compute the expression of the function to be minimized in terms of θ and compute its stationary points. Compare the obtained result with the results in part (b). [8]

4. Consider the optimization problem

$$\begin{cases} \min_{x_1,x_2} -x_1 - x_2 \\ -1 + x_1^2 + x_2^2 \leq 0 \end{cases}$$

(a) Write the first order necessary conditions and second order sufficient conditions of optimality for such a problem. Verify that all admissible points are regular points for the constraint. Hence determine candidate optimal solutions. [8]

(b) Transform the considered minimization problem into an unconstrained minimization problem using the method of the sequential penalty functions. [4]

(c) Write the necessary conditions of optimality for the unconstrained problem in part (b). Hence compute approximate candidate optimal solutions for such an unconstrained optimization problem and compare the results with those obtained in part (a). [6]

(d) Discuss the feasibility of the candidate optimal solutions computed in part (c). [2]

5. Consider the discrete time system

$$x_{k+1} = Ax_k + Bu_k$$

with $k = 0, \cdots, M$, $x \in I\!R^n$, $u_k \in I\!R$ and $x(0) = x_0$. (Note that $A \in I\!R^{n\times n}$ and $B \in I\!R^{n\times 1}$.) Assume that the square matrix

$$\mathcal{C} = \left[A^{n-1}B, \cdots, AB, B\right]$$

is invertible. Consider now the problem of determining the vector

$$U = [u_0, u_1, \cdots, u_{M-1}]'$$

such that the cost

$$J(U) = \frac{1}{2}\left(u_0^2 + u_1^2 + \cdots + u_{M-1}^2\right)$$

is minimized and the end-time condition

$$x(M) = 0$$

holds.

(a) Pose the above problem as a constrained optimization problem in the decision variables $[u_0, u_1, \cdots, u_{M-1}]$ and parameterized by x_0. [4]

(b) Show that if $M = n$ then the considered optimization problem has only one feasible solution, *i.e.* there is only one selection of $[u_0, u_1, \cdots, u_{M-1}]$ such that the end-time condition holds. [4]

(c) Suppose $M = n+1$. Show that the considered constrained optimization problem can be transformed into an unconstrained problem in the decision variable u_0. [8]

(d) Solve the unconstrained optimization problem in part (c). [4]

6. Consider the optimization problem

$$\begin{cases} \min\limits_{x_1,x_2} x_1x_2 \\ x_1^2 + x_2^2 = 1. \end{cases}$$

(a) Sketch in the (x_1, x_2)-plane the level surfaces of the function to be minimized and of the admissible set. [6]

(b) Compute an exact penalty function for the considered minimization problem. [2]

(c) Compute the stationary points and the minima of the exact penalty function constructed in part (b). Hence construct a solution of the considered constrained optimization problem. [10]

(d) Let $x^\star$ be the constrained minimum computed in part (c). Using the first order necessary conditions of optimality construct the corresponding optimal multiplier $\lambda^\star$. [2]