OPTIMIZATION — SOLUTIONS 2003

Ex 1

(a) $\dfrac{\partial f}{\partial x_1} = 6x_1^2 - 6x_1 - 12x_1x_2 + 6x_2^2 + 12x_2$

$\dfrac{\partial f}{\partial x_2} = -6x_1^2 + 12x_1x_2 - 6x_1 = -6x_1\left[x_1 - 2x_2 - 2\right]$

Stationary points: $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial f}{\partial x_2} = 0$

$\dfrac{\partial f}{\partial x_2} = 0 \rightarrow x_1 = 0 \longrightarrow 6x_2(x_2+2) = 0 \rightarrow P_1 = (0,0)$, $P_2 = (0,-2)$

$\dfrac{\partial f}{\partial x_2} = 0 \rightarrow x_1 - 2x_2 - 2 = 0$

$\downarrow$

$x_1 = 2x_2 + 2$

$\downarrow$

$(2x_2+2)^2 - (2x_2+2) - 2(2x_2+2)x_2 + x_2(x_2+2) = 0$

$\downarrow$

$x_2^2 + 4x_2 + 2 = 0 \longrightarrow x_2 = -2+\sqrt{2}$, $x_2 = -2-\sqrt{2}$

$\downarrow$

$P_3 = (-2+2\sqrt{2},\ -2+\sqrt{2})$

$P_4 = (-2-2\sqrt{2},\ -2-\sqrt{2})$

(b) $\nabla^2 f(P_1) = \begin{bmatrix} -6 & 12 \\ 12 & 0 \end{bmatrix}$ $\qquad$ $\nabla^2 f(P_2) = \begin{bmatrix} 18 & -12 \\ -12 & 0 \end{bmatrix}$

P_1 is a saddle point $\qquad$ P_2 is a saddle point

$$\nabla^2 f(P_3) \approx \begin{bmatrix} 10.9 & -4.9 \\ -4.9 & 9.9 \end{bmatrix} > 0 \quad P_3 \text{ is a local MIN}$$

$$\nabla^2 f(P_4) \approx \begin{bmatrix} -22.9 & 28.9 \\ 28.9 & -57.9 \end{bmatrix} < 0 \quad P_4 \text{ is a local MAX}$$

(c)

$$(*) \quad p_{k+1} = \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} - \alpha \begin{bmatrix} 6(x_1^k)^2 - 6x_1^k - 12x_1^k x_2^k + 6(x_2^k)^2 + 12x_2^k \\ -6x_1^k(x_1^k - 2x_2^k - 2) \end{bmatrix}$$

(d) Linear approx of (*) close to P_3

$$p_{k+1} = \begin{bmatrix} 1 + 6\alpha - 12\alpha\sqrt{2} & 12\alpha(\sqrt{2} - 1) \\ 12\alpha(\sqrt{2} - 1) & 1 + 24\alpha - 24\alpha\sqrt{2} \end{bmatrix} p_k$$

Try, e.g. $\alpha = \frac{1}{10}$,

$$p_{k+1} \approx \begin{bmatrix} -0.1 & 0.5 \\ 0.5 & 0.06 \end{bmatrix} p_k$$

$\text{Eig} = \{-0.54, 0.45\}$

The gradient algorithm defines a stable iteration for $\alpha > 0$ small and locally around a MIN.

Ex 2

(a) Consider the system of eqs

$$F(x) = 0 \quad , \; x \in \mathbb{R}^n \; , \quad F: \mathbb{R}^n \to \mathbb{R}^n$$

If the Jacobian of F exists and it is continuous then

$$F(x+s) = F(x) + \frac{\partial F}{\partial x}(x)\, s + \delta(x,s)$$

with

$$\lim_{\|s\| \to 0} \frac{\delta(x,s)}{\|s\|} = 0.$$

Hence, given x_k we compute $x_{k+1} = x_k + s$
with $s = -\left[\frac{\partial F}{\partial x}(x_k)\right]^{-1} F(x_k)$, if the inverse exists.

This yields the Newton iteration

$$x_{k+1} = x_k - \left[\frac{\partial F}{\partial x}(x_k)\right]^{-1} F(x_k)$$

(b) $F(x) = x^2 + 2bx + c$

$\frac{\partial F}{\partial x} = 2x + 2b$

$$\Longrightarrow \quad x_{k+1} = x_k - \frac{x_k^2 + 2b x_k + c}{2(x_k + b)}$$

$$\Downarrow$$

$$x_{k+1} = \frac{x_k^2 - c}{2(x_k + b)} \qquad (*)$$

Note that

$$x_{k+1} + b = \frac{x_k^2 - c}{2(x_k + b)} + b = \frac{x_k^2 - c + 2b x_k + 2b^2}{2(x_k+b)}$$

hence

$$x_{k+1} + b = \frac{\overbrace{(x_k + b)^2}^{>0} + \overbrace{b^2 - c}^{>0}}{2(x_k + b)}$$

So

$$x_k + b > 0 \longrightarrow x_{k+1} + b > 0 \quad \text{for all } k$$

$$x_k + b < 0 \longrightarrow x_k + b < 0 \quad \text{for all } k$$

(c) Set $b = 0$, $c = -3$ in (*) :

$$x_{k+1} = \frac{x_k^2 + 3}{2 x_k}$$

$x_0 = 1$

$x_1 = 2$

$x_2 = 7/4$

$x_3 = 1.73214$

$x_4 = 1.732050810$ $(\sqrt{3} = 1.732050808)$

$$\frac{x_4 - \sqrt{3}}{\sqrt{3}} = 0.1 \cdot 10^{-8}$$

Ex 3

(a) $$\mathcal{L} = x'x + 2d'x + \lambda(x'x - a^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 2d + 2\lambda x = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x'x - a^2 = 0$$

(b) $$\frac{\partial \mathcal{L}}{\partial x} = 0 \longrightarrow x = -\frac{d}{1+\lambda}$$

$$\downarrow$$

$$x'x = a^2 \longrightarrow \frac{d'd}{(1+\lambda)^2} = a^2$$

$$\longrightarrow 1+\lambda = \pm \frac{1}{a}\sqrt{d'd}$$

$$\lambda = -1 \pm \frac{1}{a}\sqrt{d'd}$$

The candidate optimal solutions are

$$x_1^* = -\frac{a\,d}{\sqrt{d'd}} \qquad x_2^* = \frac{a\,d}{\sqrt{d'd}}$$

$$(\lambda_1^* = -1 + \sqrt{d'd}/a) \qquad (\lambda_2^* = -1 - \sqrt{d'd}/a)$$

$$\nabla^2_{xx}\mathcal{L} = 2(\lambda + 1) I$$

hence x_1^* is a local min

x_2^* is a local max

(c) If $x_1 = a\cos\theta$

$x_2 = a\sin\theta$

then $x'x = x_1^2 + x_2^2 = a^2$

$x'x + 2d'x = a^2 + 2d_1\cos\theta + 2d_2\sin\theta$

Hence the constrained optimization problem is now

$$\min_{\theta} \quad a^2 + 2d_1\cos\theta + 2d_2\sin\theta = f(\theta)$$

The stationary points are such that

$$\frac{\partial f}{\partial \theta} = -2d_1\sin\theta + 2d_2\cos\theta = 0$$

$$\Downarrow$$

$$d_1\sin\theta = d_2\cos\theta$$

$$\theta = \arctan(d_2/d_1)$$

and we have two candidate solutions as in (b).

Ex 4

(a) $\mathcal{L} = -x_1 - x_2 + \rho(x_1^2 + x_2^2 - 1)$

$$\left.\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x_1} = -1 + 2\rho x_1 = 0 \\
&\frac{\partial \mathcal{L}}{\partial x_2} = -1 + 2\rho x_2 = 0 \\
&-1 + x_1^2 + x_2^2 \le 0 \\
&\rho \ge 0 \\
&\rho \cdot (-1 + x_1^2 + x_2^2) = 0
\end{aligned}\right\} \text{Nec. Cond}$$

$$\left.\begin{aligned}
&(*) \quad s' \begin{bmatrix} 2\rho & 0 \\ 0 & 2\rho \end{bmatrix} s > 0 \qquad \forall s \neq 0 \\
&\text{such that} \\
&\qquad \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} s = 0
\end{aligned}\right\} \text{Suff. Cond}$$

Note that if $\rho > 0$, then $(*)$ holds for any s, if $\rho = 0$ then $(*)$ does not hold for all s.

Note that $\nabla_x (x_1^2 + x_2^2 - 1) = [2x_1 \;,\; 2x_2]$ and this is always nonzero when $x_1^2 + x_2^2 - 1 = 0 \Rightarrow$ All points are regular.

From the Nec Cond.

either $\rho = 0 \longrightarrow -1 = 0$ False

or $\rho > 0 \longrightarrow -1 + x_1^2 + x_2^2 = 0$, i.e. all candidate solutions are on the boundary.

Note now that

$$x_1 = x_2 = \frac{1}{2\rho} \longrightarrow x_1^2 + x_2^2 = 1 = \frac{1}{4\rho^2} + \frac{1}{4\rho^2} = 1$$

$$\downarrow$$

$\rho = \pm\sqrt{1/2}$ but

$\rho > 0 \Rightarrow \rho = \sqrt{1/2}$

$$\downarrow$$

$$P_1 = (x_1, x_2) = \left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$$

P_1 is the optimal solution as also the Suff. Cond. are satisfied.

(b) Define the sequencial penalty function as

$$F_\varepsilon = -x_1 - x_2 + \frac{1}{\varepsilon}\left[\max\left(0, -1 + x_1^2 + x_2^2\right)\right]^2$$

(c) If $x_1^2 + x_2^2 - 1 \le 0 \longrightarrow F_\varepsilon = -x_1 - x_2$

$$\downarrow$$

$$\nabla F_\varepsilon = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq 0$$ No stationary points

If $x_1^2 + x_2^2 - 1 > 0 \longrightarrow F_\varepsilon = -x_1 - x_2 + \frac{(x_1^2 + x_2^2 - 1)^2}{\varepsilon}.$

$$\downarrow$$

$$\nabla F_\varepsilon = \begin{bmatrix} -1 + \frac{4(x_1^2 + x_2^2 - 1)x_1}{\varepsilon} \\ -1 + \frac{4(x_1^2 + x_2^2 - 1)x_2}{\varepsilon} \end{bmatrix}$$

$$\nabla F_\varepsilon = 0 \Longrightarrow \begin{array}{l} (x_1^2 + x_2^2 - 1)\,x_1 = \frac{\varepsilon}{4} \\ (x_1^2 + x_2^2 - 1)\,x_2 = \frac{\varepsilon}{4} \end{array} \Longrightarrow x_1 = x_2 = \xi$$

$$\Downarrow$$

$$(2\xi^2 - 1)\xi = \frac{\varepsilon}{4}$$

if ε is small

$\xi \approx 0 \qquad \xi \approx \pm\sqrt{\frac{1}{2}}$

Not in $x_1^2 + x_2^2 - 1 > 0$

We obtain two candidate optimal solutions

$$P_2 = \left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) \qquad P_3 = \left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$$

Note that as $\varepsilon \to 0$ P_2 approaches the exact optimal solution.

Ex 5

(a) Note that

$$x_1 = A x_0 + B u_0$$

$$x_2 = A x_1 + B u_1 = A^2 x_0 + A B u_0 + B u_1$$

$$\vdots$$

$$x_N = A^N x_0 + A^{N-1} B u_0 + A^{N-2} B u_1 + \cdots B u_{N-1}$$

$$\begin{cases} \min \ \frac{1}{2} \left(u_0^2 + \cdots u_{N-1}^2 \right) \\ A^N x_0 + \cdots \quad B u_{N-1} = 0 \end{cases}$$

(b) If $N = n$ then

$$x_n = x_N = A^n x_0 + \left[A^{n-1} B, \ \cdots \ B \right] \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix} = 0$$

$$= A^n x_0 + C U = 0$$

Hence the unique solution is

$$U = - C^{-1} A^n x_0 .$$

(c) If $N = n+1$ then

$$x_{n+1} = x_N = A^{n+1} x_0 + A^n B u_0 + A^{n-1} B u_1 + \cdots B u_n$$

$$= A^{n+1} x_0 + A^n B u_0 + \mathcal{C} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

hence $x_{n+1} = 0$ implies

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = -\mathcal{C}^{-1} \left[A^{n+1} x_0 + A^n B u_0 \right] = F x_0 + G u_0$$

This means that $u_1 \ldots u_n$ are functions of x_0 and u_0. The problem is not recast as

$$\min_u \frac{1}{2} \left(u_0^2 + [u_1 \cdots u_n] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \right)$$

$$\Downarrow$$

$$\min_{u_0} \frac{1}{2} \left(u_0^2 + (F x_0 + G u_0)' (F x_0 + G u_0) \right)$$

$$\Downarrow$$

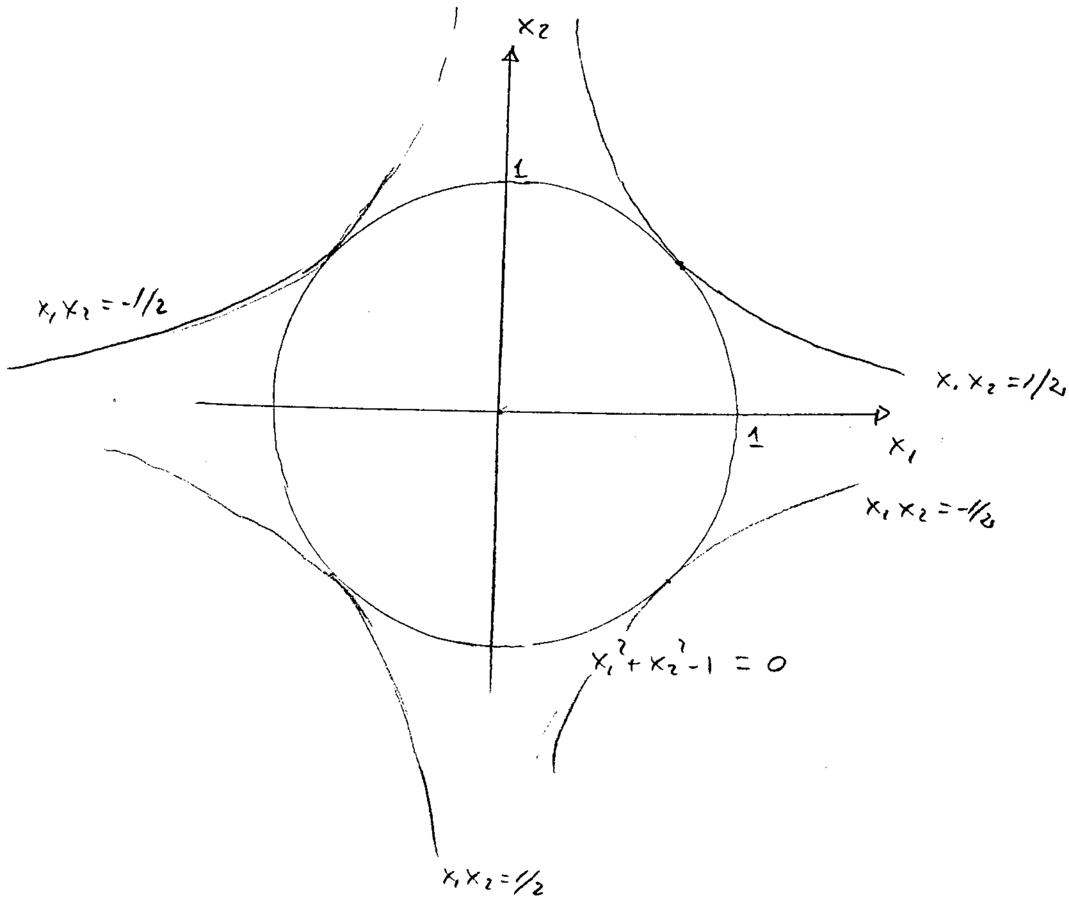
$$\min_{u_0} \frac{1}{2} \left[u_0^2 (1 + G'G) + 2 u_0 G' F x_0 + x_0^T F^T F x_0 \right]$$

(d) $$f = \frac{1}{2} \left(u_0^2 (1 + G'G) + 2 u_0 G' F x_0 + x_0^T F^T F x_0 \right)$$

$$\frac{\partial f}{\partial u_0} = u_0 (1 + G'G) + G' F x_0 \Longrightarrow u_0^* = -\frac{G' F x_0}{1 + G'G}$$

Ex 6

(a)



(b) $$\mathcal{L}_a = x_1 x_2 + \lambda (x_1^2 + x_2^2 - 1) + \frac{1}{\varepsilon} (x_1^2 + x_2^2 - 1)^2$$

with $$\lambda = -\frac{x_1 x_2}{x_1^2 + x_2^2}$$

(c) $$\frac{\partial \mathcal{L}_a}{\partial x_1} = \frac{\partial \mathcal{L}_a}{\partial x_2} = 0$$

$P_1 = \left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$

$P_2 = \left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$

$P_3 = \left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$

$P_4 = \left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$

$f(P_1) = \frac{1}{2}$ $\quad f(P_2) = -\frac{1}{2}$ $\quad f(P_3) = -\frac{1}{2}$ $\quad f(P_4) = \frac{1}{2}$

$$\nabla^2 L_a(P_2) = \begin{bmatrix} 1 + 4/\varepsilon & 1 - 4/\varepsilon \\ 1 - 4/\varepsilon & 1 + 4/\varepsilon \end{bmatrix} > 0$$

$$\nabla^2 L_a(P_3) = \nabla^2 L_a(P_2) > 0$$

P_2 and P_3 are local minima.

(d) $$\mathcal{L} = x_1 x_2 + \lambda(x_1^2 + x_2^2 - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 + 2\lambda x_1 \longrightarrow \lambda^* = -\frac{x_2}{2x_1} = \frac{1}{2}$$